International Journal of Science and Research Archive
eISSN: 2582-8185
Cross Ref DOI: 10.30574/ijsra
Journal homepage: https://ijsra.net/

(Research Article)

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Statefinder and O_m diagnostic parameter along with Energy Conditions of one-dimensional Bulk viscous cosmic string in f (R, T) theory of gravity

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International Journal of Science and Research Archive, 2025, 16(03), 284–301

Publication history: Received on 22 July 2025; revised on 03 September 2025; accepted on 05 September 2025

Article DOI: https://doi.org/10.30574/ijsra.2025.16.3.2504

Abstract

A spatially homogeneous and anisotropic Bianchi type VI_0 cosmological model is considered when the source for energy momentum tensor is a bulk viscous fluid containing one dimensional cosmic string in the framework of f (R, T) theory of gravity. We solve field equations using three different cases: (i) constant deceleration parameter. (ii) the expansion scalar of the space-time is proportional to the shear scalar (iii) A barotropic equation of state. Also, the bulk viscous pressure is assumed to be proportional to the energy density. The physical behavior of the investigated model is discussed and also the graphical behavior of parameters is discussed. Also, we analyzed jerk parameter and the state finder parameter. When we work on the $O_m(z)$diagnostic parameter we get negative slope and found that the model exhibited quintessence-like behavior. Our investigation concludes that the proposed cosmological model is in strong agreement with recent observational studies and successfully explains the late-time cosmic acceleration. Our study demonstrates that the derived cosmological model satisfies all energy conditions, indicating its physical plausibility without any violations.

Keywords: f (R, T) Gravity; O_m Diagnostics Parameter; Bulk Viscous Fluid; One Dimensional Cosmic String; Energy Conditions

1. Introduction

Astronomical observations have revealed that our Universe is currently undergoing an accelerated expansion. This phenomenon was first detected through high-redshift Type Ia supernovae [1-6] and later confirmed by cosmic microwave background radiation [7,8] and large-scale structure studies [9-12]. To explain this acceleration, an exotic component with significant negative pressure, known as Dark Energy (DE), has been proposed. Current estimates suggest that our Universe comprises approximately one-third dark matter and two-thirds dark energy. Various theoretical models have been introduced to describe the nature of DE and dark matter, including quintessence scalar field models [13,14], K-essence [15,16], phantom fields [17,18], tachyon fields [19,20] and quintom models [21,22], among others. Additionally, alternative theories of gravity have been explored in DE models by researchers such as Pawar et al. [23], Samanta [24,25], and Pawar and Solanke [26]. Einstein's theory of gravitation has been instrumental in explaining the evolution and origin of the Universe, as well as in the development of cosmological models. However, it falls short in accounting for late-time cosmic acceleration. To address this limitation, numerous modifications to gravity have been proposed. One such approach involves replacing the standard gravitational action with an arbitrary function of the Ricci scalar, leading to the development of $f(R)$ gravity [27]. A comprehensive review of $f(R)$ gravity was provided by Copeland et al. [28], while Chiba et al. [29] explored various $f(R)$ gravity models that successfully describe both late-time acceleration and early Universe inflation.

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Among the notable extensions of modified gravity, $f(R,T)$ gravity, introduced by Harko et al. [30], considers the gravitational Lagrangian as a function of the Ricci scalar (R) and the trace of the stress-energy tensor (T). By varying the gravitational action with respect to the metric tensor, the field equations of this theory can be derived. Further investigations into $f(R,T)$ gravity have been carried out by Samanta and Myrzakulov [31], who analyzed cosmological models with imperfect fluids. Venkateswarlu et. al. [32] have investigated Bianchi type-I, II, VIII &IX string cosmological solutions in self-creation theory of gravitation. Recently Rao et. al. [33] have obtained exact Bianchi type-II, VIII & IX string cosmological models. Additionally, several researchers, including Sharif [34], Keskin [35], Reddy et al. [36,37], and Chaubey and Shukla [38], have explored cosmological models within the framework of $f(R,T)$ gravity using different Bianchi-type space-times.

The study of cosmic string has received considerable attention in cosmology as they play important role in the structure formation and evolution of universe. The cosmic strings have their stress-energy coupled to the gravitational field. Therefore, it will be interesting to study of gravitational effects of such strings. The gravitational effects of cosmic strings in general relativity have been extensively studied by Kibble et.al.,1982; Vilenkin,1981; Goetz, 1990; Letelier 1983; Stachel, 1988 etc. [39-43] There is no direct evidence of strings in the present-day universe, but the cosmological model of the universe which evolve from a string dominated era and end up in a particle dominated era are of physical interest. In 2013 Kiran et.al. [44] have shown that in Bianchi type-III space time bulk viscous string cosmological model does not exist under the framework of $f(R,T)$ gravity. Naidu et.al. in 2013 and Reddy et.al. in 2014 [45,46] studied the cosmic strings with $f(R,T)$ gravity theory and they found out the non-existence of cosmic string and perfect fluid in this particular theory. In 2011, D Pawar et.al.[47] investigated the "string cosmological model in presence of massless scalar Field in modified theory of general relativity." and in 2023 K Pawar et.al. [48] have investigated the Bianchi type VI_0 space-time in the presence of string of clouds coupled with perfect fluid within the context of $f(R,T)$ gravity in 2023. Bayaskar et.al. [49] have analyzed "Plane symmetric cosmological model of interacting fields in general relativity" in 2009. Pawar et.al. [50] have found out "Plane Symmetric Cosmological Model with Thick Domain Walls in Brans-Dicke Theory of Gravitation." Recently in 2023 Mete et.al. [51] have discussed "Five-Dimensional Cosmological Model with One Dimensional Cosmic String Coupled with Zero Mass Scalar Field in Lyra Manifold." And in 2025 Bayaskar et.al. studied "Bianchi Type VI_0 One Dimensional Cosmic String for Bulk Viscous Fluid in $f(R,T)$Theory of Gravity." [52]. Chirde et al. in 2020 [53] investigated the LRS Bianchi type I metric with the source as barotropic perfect fluid and cosmic string in the framework of $f(T)$ gravity using three different functional forms of $f(T)$ gravity. Bhoyar et.al. in 2019 [54] studied Kantowaski-Sachs cosmological model with viscous cosmic string in the quadratic form of teleparallel gravity for a particular choice of $f(T)$ formalism. Ujjal Debnath et.al. studied Statefinder and O_m Diagnostics for Interacting New Holographic Dark Energy Model [55]. M. Shahalam, et.al have investigated "O_m diagnostic applied to scalar field models and slowing down of cosmic acceleration" in 2015 [56].

Various viewpoints exist concerning the Universe, and a definitive conclusion has yet to be reached. The history and evolution of the cosmos remain topics of ongoing debate. Thus, it is our duty to continue exploring and investigating the mysteries of cosmology, including the many elusive particles that are yet to be discovered. Building on the above motivations, this research paper focuses on emphasizing the study of Bianchi type VI_0 space-time with one dimensional cosmic string for a bulk viscous fluid in the context of $f(R,T)$ gravity. The physical and geometrical aspects of the models are also studied with their graphical behaviour. Along with the physical aspects we discussed the energy conditions. Statefinder and O_m diagnostics also discussed to distinguishing ΛCDM from alternative dark energy models. This paper is organized as follows: Section 2 provides the brief methodology of $f(R,T)$ gravity. In section 3 the metric and derivation of its field equations is given. After that in the next section the solution of the field equation is discussed. In section 5 Physical and Kinematical Properties of the Model are discussed. Statefinder parameter is discussed in Section 6. In section 7 O_m diagnostics parameter is given and in section 8 Energy conditions are studied. In last section (section 9) we conclude the present work.

2. Methodology of $f(R,T)$ theory of gravity

In 2011 Harko et. al. proposed another modification of Einstein's theory of gravitation which is known as $f(R,T)$theory of gravity. Wherein the gravitational Varangian is given by an arbitrary function of the Ricci scalar R and of the trace T of the stress energy tensor T_{ij}.

We can obtain the field equation of $f(R,T)$gravity by using the Hilbert-Einstein action, we take the total action of $f(R,T)$ gravity given by Harko [30] as,

$$S = \int \left[\frac{1}{2k} f(R,T) + L_m\right] \sqrt{-g}\ d^4x \tag{1}$$

Where, $k = 8\pi G$, g is the determinant of the metric, L_m is the matter Lagrangian density, $f(R,T)$ is the arbitrary function of the scalar curvature Rand the trace T of the energy momentum tensor T_{ij} .

We take the energy momentum tensor T_{ij} of the matter source as,

$$T_{ij} = \frac{-2}{\sqrt{-g}} - \frac{\delta(\sqrt{-g}\, L_m)}{\delta\, g^{ij}} \quad (2)$$

Where its trace is given by $T = g^{ij}T_{ij}$

By differentiating the action (1) of the gravitational field with respect to the metric tensor g_{ij}, we get the field equation for $f(R,T)$ gravity as,

$$f_R(R,T)R_{ij} - \frac{1}{2} f(R,T)g_{ij} + (g_{ij}\Box - \nabla_i\nabla_j)f_R(R,T) = kT_{ij} - f_T(R,T)T_{ij} - f_T(R,T)\theta_{ij} \quad (3)$$

where, $f_R(R,T) = \frac{\partial f(R,T)}{\partial R}$, $f_T(R,T) = \frac{\partial f(R,T)}{\partial T}$, $\theta_{ij} = g^{mn}\frac{\delta T_{mn}}{\delta g^{ij}}$, $\Box = \nabla^i\nabla_j$

$\nabla_i\nabla_j$ denotes the covariant derivative.

Here c is the speed light in vacuum and the Newtonian gravitational constant is G

Now we consider the stress energy tensor of the matter as,

$$T_{ij} = (\rho + \bar{p})u_i u_j - \bar{p}\, g_{ij} \quad (4)$$

This expression is a consequence of a statement of fluid mechanics for which we get a relation between matter Lagrangian density L_m and pressure of the fluid as L_m = - p.

$$u^i\nabla_j u_i = 0\,,\ u^i u_i = 1 \quad (5)$$

By using the above value of θ_{ij} , we get the variation of stress energy of perfect fluid as,

$$\theta_{ij} = -2T_{ij} - \bar{p}g_{ij} \quad (6)$$

The different forms of matter distribution will yield different theoretical models of $f(R,T)$ gravity However, [30] have obtained three particular classes of $f(R,T)$ gravity models as

$$f(R,T) = \begin{Bmatrix} R + 2f(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R)f_3(T) \end{Bmatrix} \quad (7)$$

Here we take,

$$f(R,T) = R + 2f(T) \quad (8)$$

For this particular choice of $f(R,T)$, from (3), we get the field equation as

$$R_{ij} - \frac{1}{2} R\, g_{ij} = 8\pi T_{ij} - 2f'(T)\, T_{ij} - 2f'(T)\, \theta_{ij} + f(T)g_{ij} \quad (9)$$

The value of θ_{ij} given in equation (6) and above equation gives us a field equation in the form

$$R_{ij} - \frac{1}{2} R\, g_{ij} = 8\pi T_{ij} + 2f'(T)\, T_{ij} + 2\bar{p}f'(T)\, g_{ij} + f(T)g_{ij} \quad (10)$$

Where $f'(T)$ is the differentiation of $f(T)$ with respect to the argument T.

3. Metric and Field Equation

Here we assume that the geometry of the universe is described by the spatially homogeneous and anisotropic Bianchi type VI_0 space-time is stated by the metric,

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 e^{-2lx} dy^2 + C^2 e^{2lx} dz^2 \quad (11)$$

where A, B, and C are the functions of cosmic time t and l is an arbitrary constant.

Now, we consider the energy momentum tensor which contains one dimensional cosmic string for a bulk viscous fluid is taken as

$$T_{ij} = (\rho + \bar{p})\, u_i u_j + \bar{p}\, g_{ij} - \lambda\, x_i x_j \quad (12)$$

And $\bar{p} = p - 3\,\zeta\, H$ (13)

where, ρ is the rest energy density of the system, $\zeta(t)$ is the coefficient of bulk viscosity, H is the Hubble's parameter, $3\,\zeta\, H$ is generally known as bulk viscous pressure. We consider $\bar{p}, \rho, and\ \lambda$ are the time dependent functions only.

Here the four-velocity vector is $u^i = \delta_4^i$ and x^i the unit space like vector represents a direction of anisotropy, i.e. the direction of string satisfies the relation given by

$$g_{ij} u^i u_j = -x^i x_j = -1 \quad (14)$$

And $u^i x_i = 0$ (15)

Using comoving coordinates system and a particular choice of the function given by Harko [30] we take the function f (T) as,

f (T) = μ T (16)

where, μ is constant.

Now, by assuming the commoving coordinate system, the field equation (10) for the metric given by (11) using the equations (12) – (14) the field equations become

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{l^2}{A^2} = \bar{p}\,(8\pi + 7\mu) - \mu\rho - \mu\lambda \quad (17)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{l^2}{A^2} = \bar{p}\,(8\pi + 7\mu) - \mu\rho - \mu\lambda \quad (18)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{B}\dot{A}}{BA} - \frac{l^2}{A^2} = \bar{p}\,(8\pi + 7\mu) - \lambda(8\pi + 3\mu) - \mu\rho \quad (19)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{l^2}{A^2} = -\rho\,(8\pi + 3\mu) + 5\bar{p}\mu - \mu\lambda \quad (20)$$

$$\frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0 \quad (21)$$

An overhead dot (·) shows the derivative with respect to cosmic time t.

For solving the field equations (17) – (21), let us define some other cosmologically important parameters, which are also helpful for the physical and kinematical analysis of the solution for the space time given by (11).

The average scale factor $a(t)$ of the Bianchi type VI0 space-time is defined as

$$a(t) = (ABC)^{\frac{1}{3}} \quad (22)$$

The spatial volume V of the metric is given by

$$V = a^3(t) = ABC \tag{23}$$

The directional Hubble parameter is given by

$$H_1 = \frac{\dot{A}}{A} \quad H_2 = \frac{\dot{B}}{B} \quad H_3 = \frac{\dot{C}}{C} \tag{24}$$

The average Hubble parameter H is given by

$$H = \frac{1}{3}(H_1 + H_2 + H_3) = \frac{1}{3}(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) \tag{25}$$

The scalar expansion θ is given by

$$\theta = 3H = (\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) \tag{26}$$

The Shear scalar σ2 is given by

$$\sigma^2 = \frac{1}{2}\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right) - \frac{1}{6}\theta^2 \tag{27}$$

The mean anisotropic parameter Δ is given by

$$\Delta = \frac{1}{3} \sum_{i=1}^{3}(\frac{H_i - H}{H})^2 \tag{28}$$

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4. Solutions of the Field Equation

After integrating equation (21), we get

$$B = \gamma . C \tag{29}$$

Where γ is the integration constant. Without loss of simplification we take it as unity, therefore equation (29) becomes

$$B = C \tag{30}$$

Solving equations (17)-(20) we get

$$\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{A}\dot{C}}{AC} + \frac{2l^2}{A^2} = 0 \tag{31}$$

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{C}}{C} - \frac{\dot{B}\dot{C}}{BC} - \frac{2l^2}{A^2} = -\lambda(8\pi + 2\mu) \tag{32}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{l^2}{A^2} = -\rho\,(8\pi + 3\mu) + 5\bar{p}\mu - \mu\lambda \tag{33}$$

Here we have system of three equations with six unknown parameters $A, B, C, p, \rho, \lambda$.

Hence, we required three additional constraints which relates these parameters to obtain explicit solutions of the system of equation. We use the following possible physical conditions.

Variation of Hubble's parameter proposed by Berman [57] that yields constant deceleration parameter models of the universe is defined by,

$$q = \frac{-a\ddot{a}}{\dot{a}^2} = \text{constant} \tag{34}$$

Here, $a = a(t)$ is the average scale factor.

The shear scalar σ is proportional to scalar expansion θ of the space time so that we can take [58]

$$A = B^{m} \tag{35}$$

Where $m \neq 0$ is a positive constant.

For a barotropic fluid the combined effect of the bulk viscous pressure and the proper pressure can be expressed as

$$\bar{p} = p - 3\zeta H = \varepsilon\rho \quad , \; p = \varepsilon_0\rho \tag{36}$$

where $\varepsilon = \varepsilon_0\gamma \quad (0 \le \varepsilon_0 \le 1)$ and $\varepsilon , \varepsilon_0 , \gamma$ are constants.

Now equation (34) admits the following solution

$$a(t) = (\alpha t + \beta)^{\frac{1}{1+q}} \tag{37}$$

Where, α (≠ 0) & β are constant of integration. This equation implies that the condition for expansion of the universe is 1+q > 0.

Now from (22) (30) (35) (37) gives the metric potential as,

$$A = (\alpha t + \beta)^{\frac{3m}{(1+q)(m+2)}} \qquad B = C = (\alpha t + \beta)^{\frac{3}{(1+q)(m+2)}} \tag{38}$$

where, $q \neq -1 , m \neq -2$.

Using (38) and by suitable choice of coordinates and constants i.e. taking $\alpha = 1, \beta = 0$,

The metric given by (11) can be written as,

$$ds^2 = -dt^2 + t^{\frac{3m}{(1+q)(m+2)}}dx^2 + t^{\frac{3}{(1+q)(m+2)}}\left(e^{-2lx}dy^2 + e^{2lx}dz^2\right) \tag{39}$$

5. Some Physical and Kinematical Properties of the Model.

Equation (39) represents the anisotropic Bianchi type VI_0 bulk viscous string cosmological model in $f(R,T)$ theory of gravitation with the following expression for the physical and kinematical parameters which are significant in the physical discussion of the cosmological model. Some physical parameters are,

The average scale factor $a(t)$ of the Bianchi type VI_0 space-time is defined as

$$a(t) = (ABC)^{\frac{1}{3}} = t^{\frac{1}{1+q}} \tag{40}$$

The spatial volume V of the metric is given by

$$V = a^3(t) = t^{\frac{3}{1+q}} \tag{41}$$

The directional Hubble parameter is given by

$$H_1 = \frac{1}{(1+q)\,t} \qquad H_2 = \frac{1}{(1+q)\,t} \quad H_3 = \frac{1}{(1+q)\,t} \tag{42}$$

The average Hubble parameter H is given by

$$H = \frac{1}{(1+q)\,t} \qquad \& \quad H(z) = H_0\left[(1+z)^{(1+q)}\right] \tag{43}$$

The scalar expansion θ is given by

$$\theta = \frac{3}{(1+q)\, t} \qquad \& \qquad \theta(z) = \frac{3(1+z)^{(1+q)}}{(1+q)} \tag{44}$$

The Shear scalar σ2 is given by

$$\sigma^2 = \frac{3(m-1)^2}{[(1+q)(m+2)t]^2} \qquad \& \qquad \sigma^2(z) = \frac{3(m-1)^2(1+z)^{2(1+q)}}{[(1+q)(m+2)]^2} \tag{45}$$

The mean anisotropic parameter Δ is given by

$$\Delta = \frac{2(m-1)^2}{(m+2)^2} \tag{46}$$

Using equation (30), (31), (32) we get

$$\lambda = 0 \tag{47}$$

Putting the value of A, B, C in (33) and using (36) & (47) we get the equations

$$\rho = \frac{1}{8\,\pi+\mu\,(3-5\varepsilon)}\left[\frac{l^2}{t^{\frac{2}{(1+q)}}} - \frac{3}{(1+q)^2\, t^2}\right] \tag{48}$$

$$p = \frac{\varepsilon_0}{8\,\pi+\mu\,(3-5\varepsilon)}\left[\frac{l^2}{t^{\frac{2}{(1+q)}}} - \frac{3}{(1+q)^2\, t^2}\right] \tag{49}$$

$$\zeta = \frac{\varepsilon_0-\varepsilon}{3\,[8\,\pi+\mu\,(3-5\varepsilon)]}\left[l^2\,(1+q)\, t^{\frac{-1+q}{1+q}} - \frac{3}{(1+q)\, t}\right] \tag{50}$$

5.1. Jerk Parameter

In cosmology, the jerk parameter is a dimensionless measure that quantifies the variation in the acceleration of the Universe's expansion over time. It is a higher-order derivative of the scale factor a(t), which measures the rate of expansion. The jerk parameter is particularly useful in studying the nature and behavior of dark energy—the enigmatic force driving the accelerated expansion of the Universe. By analyzing the jerk parameter, scientists can gain insights into whether the Universe's expansion is transitioning between phases, such as shifting from acceleration to deceleration or vice versa, offering clues about the long-term fate of the cosmos. The jerk parameter(j), can be defined as follows:

$$j = \frac{\dddot{a}}{aH^3} \tag{51}$$

By using equation (20) and (21) in above equation we get,

$$j = q + 2q^2 \tag{52}$$

5.2. Statefinder Parameter

It provides a model-independent approach to characterizing the intrinsic properties of dark energy through higher derivatives of the scale factor. By utilizing the cosmic statefinder diagnostic pair {r, s}, this method enables researchers to explore the nature of dark energy without relying on specific models. The framework also highlights subtle differences in dark energy behavior, offering deeper insights into its nature and its role in the Universe’s accelerated expansion. The statefinder parameters are usually expressed as:

$$r = \frac{\dddot{a}}{aH^3} \qquad \text{and} \qquad s = \frac{r-1}{3\left(q-\frac{1}{2}\right)} \tag{53}$$

Distinguishing between different cosmological domains largely depends on the trajectories within the {r, s} plane. Each point on this plane corresponds to a distinct cosmological model, offering a visual representation of their unique properties.

- (r = 1, s = 0): ΛCDM
- (r = 1, s = 1): SCDM
- (r = 1, s = $\frac{2}{3}$): Holographic dark energy
- (r > 1, s < 0): Phantom region
- (r < 1, s > 0): Quintessence region

By using equation (20) and (21) in above equation we get,

$$r = q + 2q^2 \quad \text{and} \quad s = \frac{2}{3}(q + 1) \tag{54}$$

We can see that for a given model, q is constant, therefore r and s are also constant. For different values of q, the corresponding expansion factors are presented in the following table for analysis.

Table 1 Statefinder pair for different values of q

Type of model	**q**	**r**	**s**
ΛCDM	-1	1	0
SCDM	0.5	1	1
Holographic dark energy	0	1	$\frac{2}{3}$
Phantom region	-2	6	$-\frac{2}{3}$
Quintessence region	-0.5	0	$\frac{1}{3}$

5.3. $O_m(z)$ diagnostic parameter:

The O_m diagnostic, combining the Hubble parameter and redshift, introduced by Sahni [59] serves as a valuable tool to distinguish between dark energy models and the standard ΛCDM framework. It is a model-independent diagnostic, making it particularly useful for distinguishing ΛCDM from alternative dark energy models.

A constant $O_m(z)$ indicates that dark energy is a cosmological constant, while a positive slope suggests phantom ($\omega < -1$) behaviour and negative slope suggests quintessence ($\omega > -1$) behavior respectively. The $O_m(z)$ diagnostic, in parallel with statefinder parameters (r, s), offers a robust framework for understanding the nature of dark energy and the cosmic expansion history. This diagnostic is particularly effective in probing the present matter density contrast and the evolution of dark energy. The $O_m(z)$ parameter is defined as:

$$O_m(z) = \frac{\left[\frac{H(z)}{H_0}\right]^2 - 1}{(1+z)^3 - 1} \tag{55}$$

where $H(z)$ is the Hubble parameter at redshift z, and H_0 is the current Hubble constant. From equation (43), (55) becomes,

$$O_m(z) = \frac{(1+z)^{2(1+q)} - 1}{(1+z)^3 - 1} \tag{56}$$

5.4. Energy Conditions

Energy conditions serve as a bridge between theoretical models and observational constraints, helping to test the viability of new ideas in cosmology. These conditions are used to explore the validity of various cosmological models, including those involving dark energy and modified theories of gravity. They are based on the Einstein field equations and help ensure physically meaningful solutions. These conditions provide insights into the nature of spacetime and its interaction with energy-matter distributions. Violations of specific energy conditions, such as the SEC, are linked to phenomena like the Universe's accelerated expansion and the existence of exotic matter. In this section, we analyze the energy conditions of our investigated model and We also evaluate whether our model meets the requirements of the energy conditions or not. These are four primary energy conditions:

5.4.1. Strong Energy Condition (SEC): $\boldsymbol{\rho + 3p \geq 0}$

The SEC implies that the total energy density, including contributions from pressure, contributes positively to the gravitational attraction. It is often violated in the presence of dark energy.

$$\rho + 3p = (1 + 3\varepsilon_0)\,\frac{1}{8\,\pi + \mu\,(3 - 5\varepsilon)}\left[\frac{l^2}{t^{\frac{2}{(1+q)}}} - \frac{3}{(1+q)^2\,t^2}\right] \geq 0 \tag{57}$$

5.4.2. Null Energy Condition (NEC): $\boldsymbol{\rho + p \geq 0}$

This condition ensures that the energy density and pressure combined are non-negative along null trajectories, such as light rays.

$$\rho + p = (1 + \varepsilon_0)\,\frac{1}{8\,\pi + \mu\,(3 - 5\varepsilon)}\left[\frac{l^2}{t^{\frac{2}{(1+q)}}} - \frac{3}{(1+q)^2\,t^2}\right] \geq 0 \tag{58}$$

5.4.3. Weak Energy Condition (WEC): $\boldsymbol{\rho \geq 0,\ \rho + p \geq 0}$

The WEC requires that the energy density is non-negative for any observer and that the effective energy density along a time like trajectory is also non-negative

$$\rho = \frac{1}{8\,\pi + \mu\,(3 - 5\varepsilon)}\left[\frac{l^2}{t^{\frac{2}{(1+q)}}} - \frac{3}{(1+q)^2\,t^2}\right] \geq 0 \quad \&$$

$$\rho + p = (1 + \varepsilon_0)\,\frac{1}{8\,\pi + \mu\,(3 - 5\varepsilon)}\left[\frac{l^2}{t^{\frac{2}{(1+q)}}} - \frac{3}{(1+q)^2\,t^2}\right] \geq 0 \tag{59}$$

5.4.4. Dominant Energy Condition (DEC): If $\boldsymbol{\rho \geq 0, \rho \geq |p|}$

This condition ensures that the energy density is greater than or equal to the magnitude of the pressure, maintaining a causal flow of energy (no superluminal energy propagation).

$$\rho + p = (1 + \varepsilon_0)\,\frac{1}{8\,\pi + \mu\,(3 - 5\varepsilon)}\left[\frac{l^2}{t^{\frac{2}{(1+q)}}} - \frac{3}{(1+q)^2\,t^2}\right] \geq 0$$

$$\rho - p = (1 - \varepsilon_0)\,\frac{1}{8\,\pi + \mu\,(3 - 5\varepsilon)}\left[\frac{l^2}{t^{\frac{2}{(1+q)}}} - \frac{3}{(1+q)^2\,t^2}\right] \geq 0 \tag{60}$$

These conditions provide important insights into the feasibility of different cosmological models and their capacity to explain the observed accelerated expansion of the Universe.

5.5. Graphical Interpretation

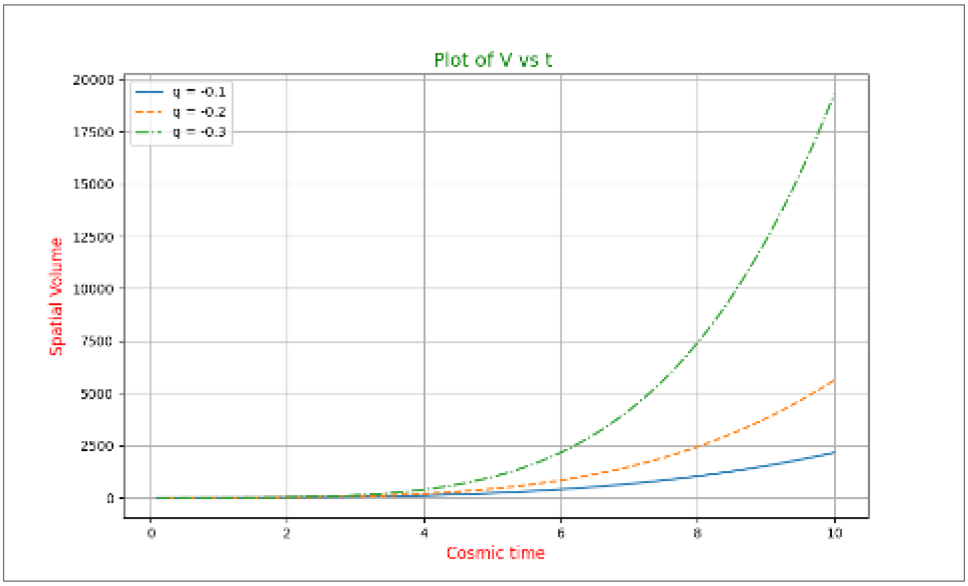

Figure 1 Variations of spatial volume with respect to cosmic time

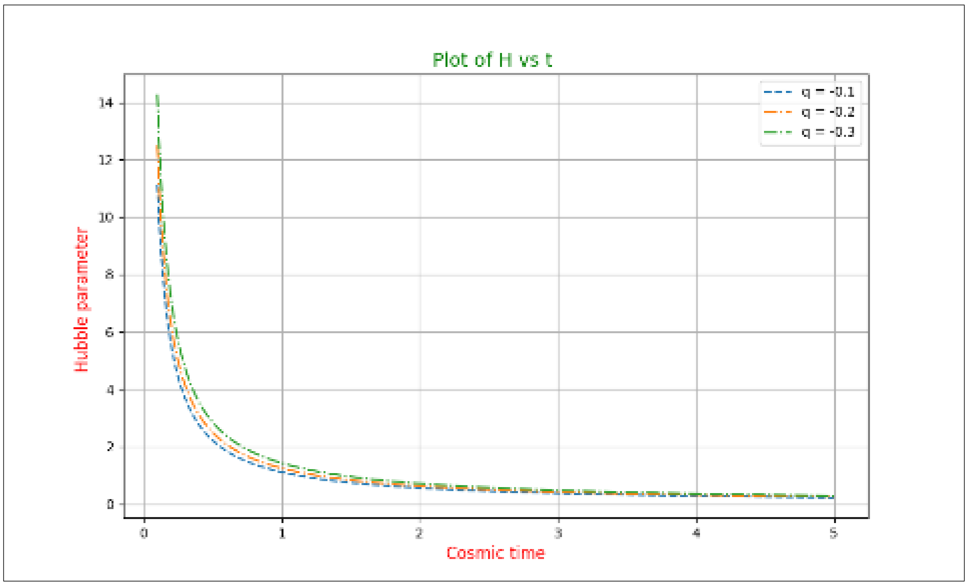

Figure 2 Variations of Average Hubble Parameter with respect to cosmic time

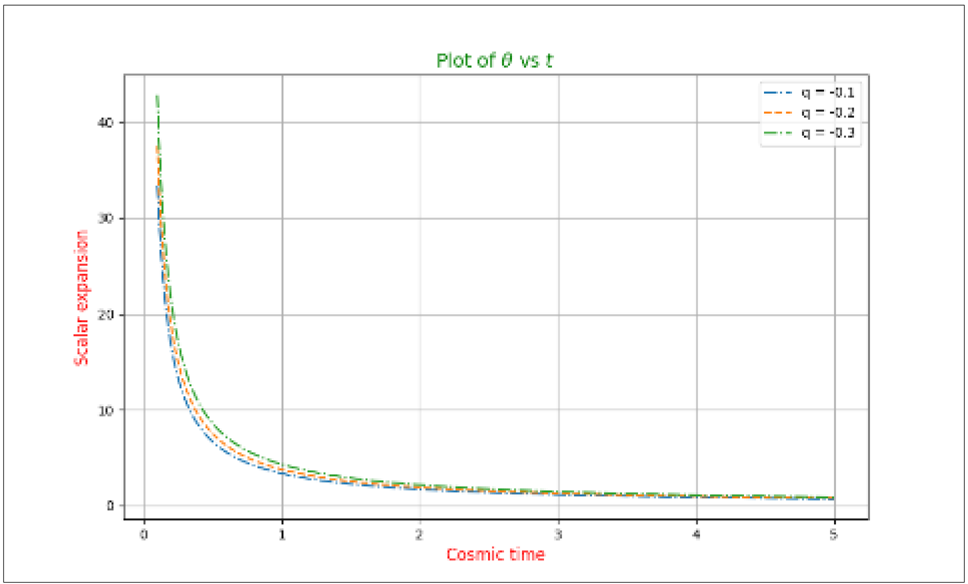

Figure 3 Variations of Scalar expansion with respect to cosmic time

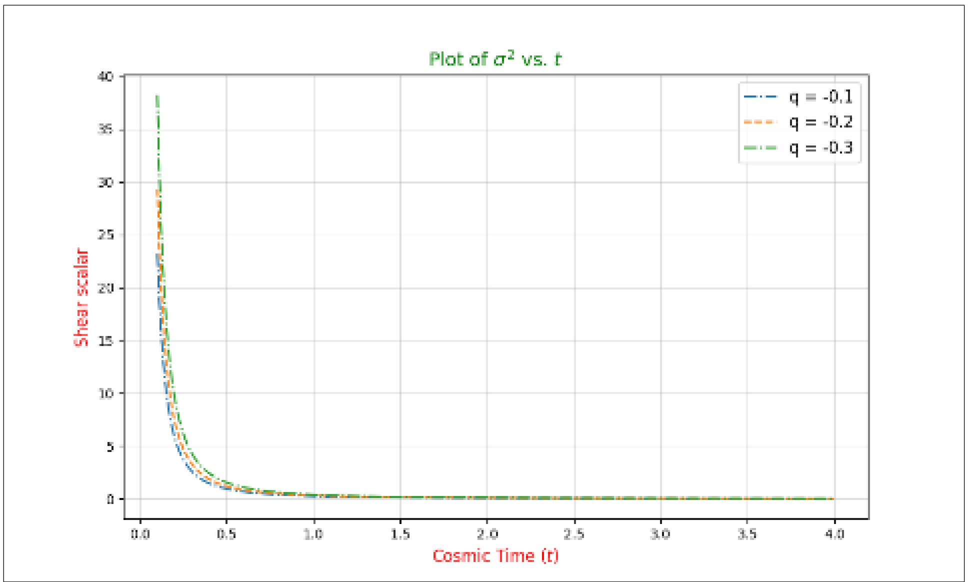

Figure 4 Variations of Shear scalar with respect to cosmic time

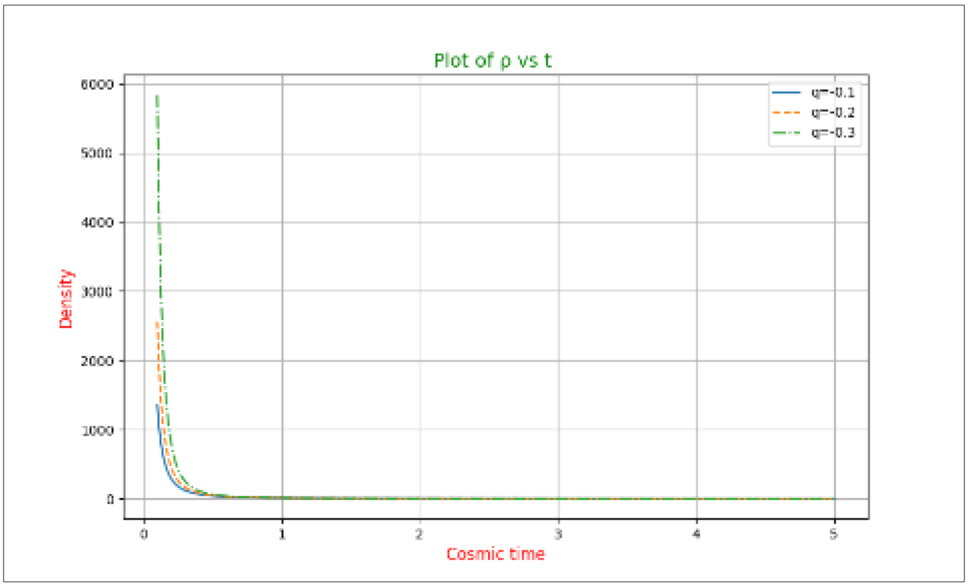

Figure 5 Variations of Density with respect to cosmic time. All terms have arbitrary units

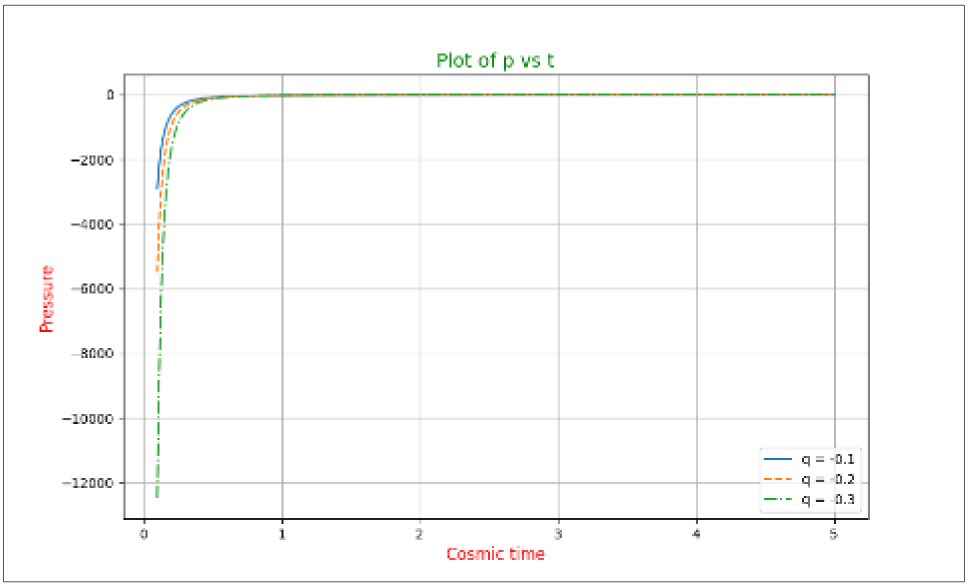

Figure 6 Variations of Pressure with respect to cosmic time. All terms have arbitrary units

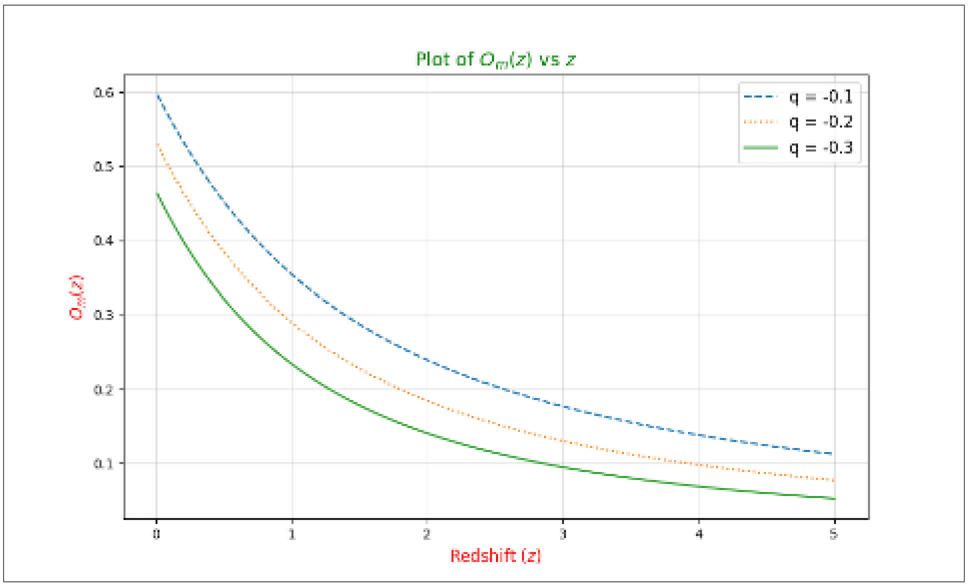

Figure 7 Variations of $\boldsymbol{O_m(z)}$ with respect to Redshift

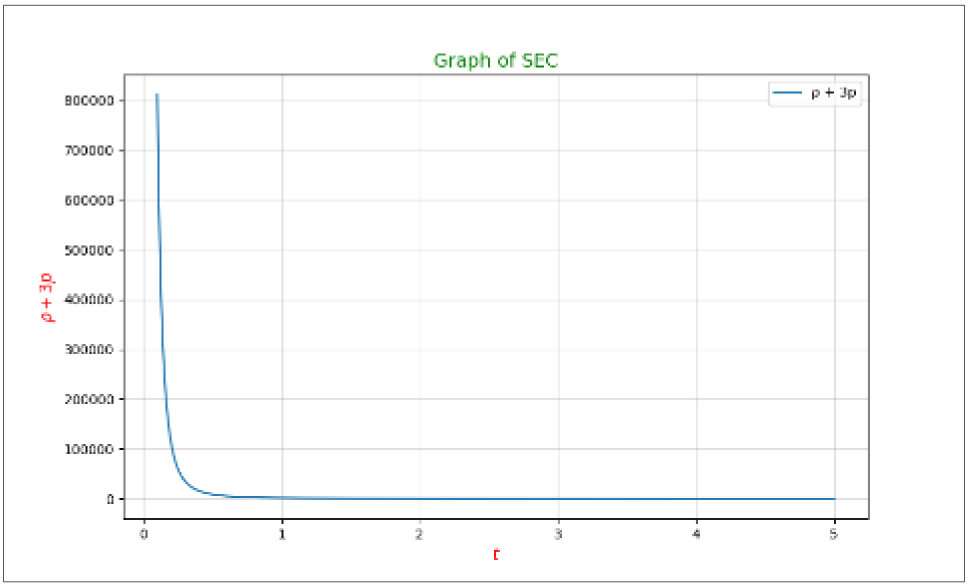

Figure 8 Variations of $\boldsymbol{\rho + 3p}$ with respect to cosmic time. (SEC)

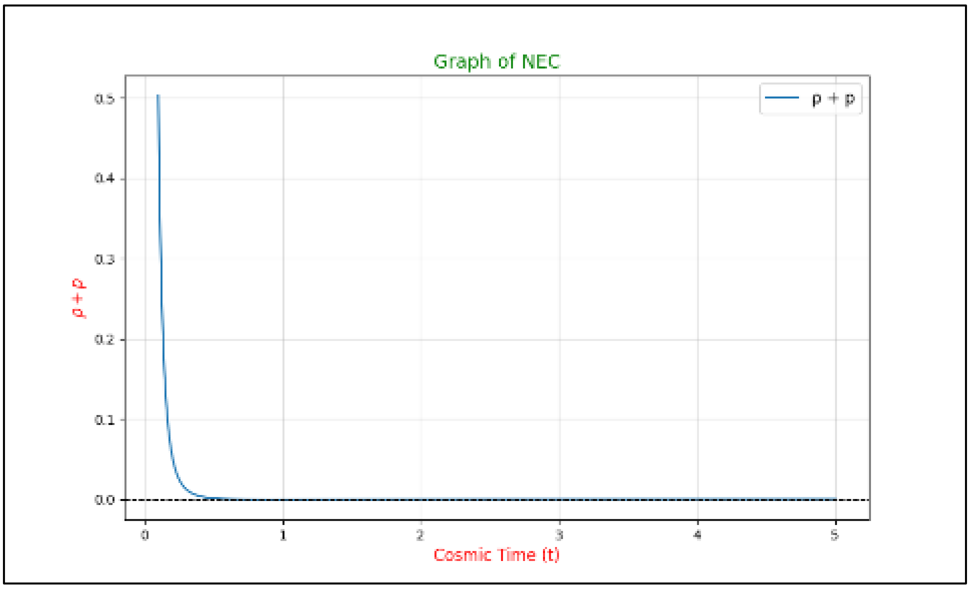

Figure 9 Variations of $\rho + p$ with respect to cosmic time. (NEC)

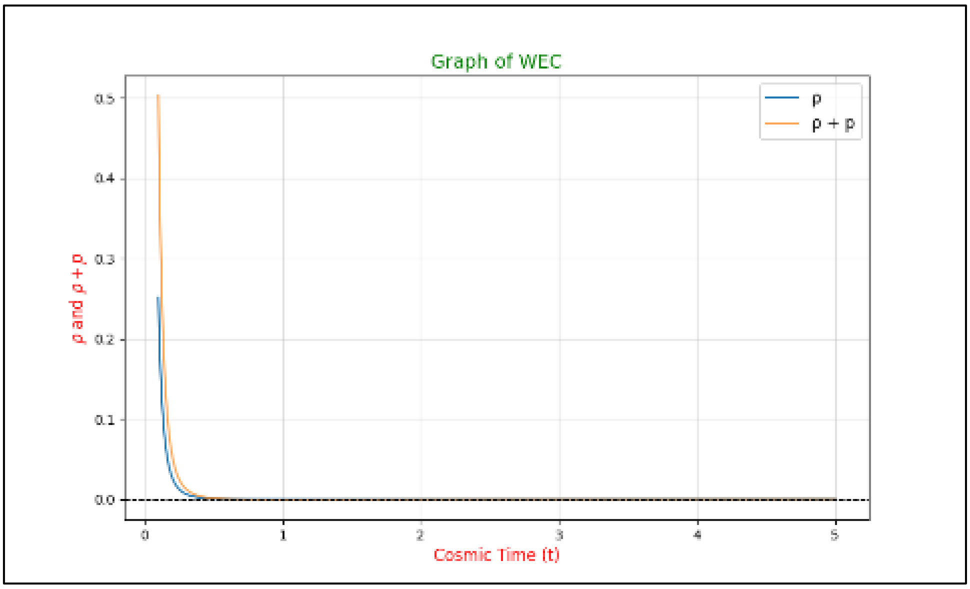

Figure 10 Variations of ρ & $\rho + p$ with respect to cosmic time. (WEC)

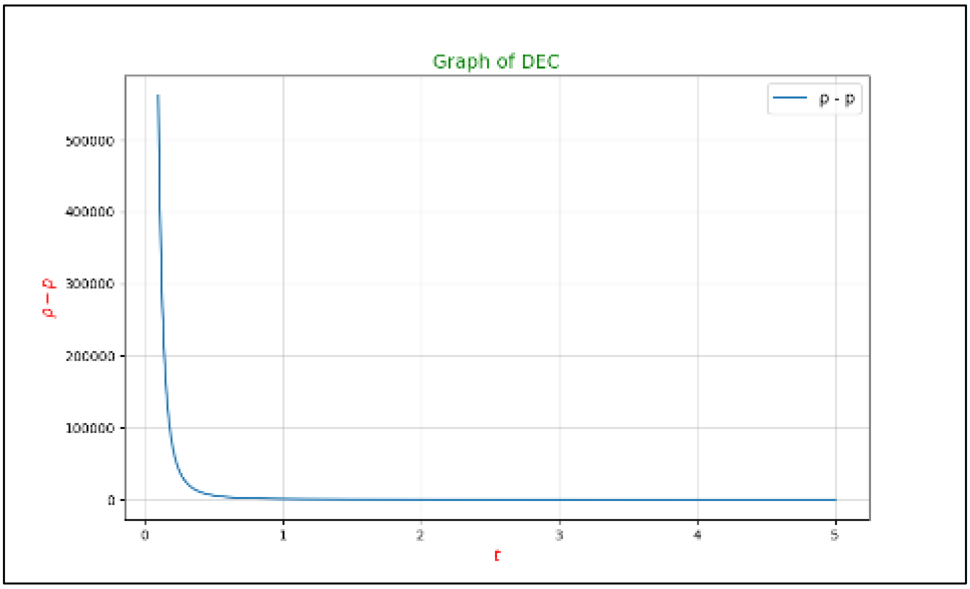

Figure 11 Variations of $\boldsymbol{\rho - p}$ with respect to cosmic time. (DEC)

From the figure we observe that,

- The increasing trend of the Universe's volume, as illustrated in Figure 1, serves as a strong indicator of cosmic expansion—an essential concept in our understanding of the Universe. These findings align with observational data, supporting the Big Bang theory and the concept of Universe undergoing perpetual expansion.
- The Hubble parameter is observed to be a decreasing function of cosmic time in the positive region (Figure 2), confirming the Universe's expansion. And, the scalar expansion initially exhibits a rapid rate, which gradually slows down over time and eventually approaches zero for large (t) (Figure 3). This behavior signifies the dynamic nature of cosmic evolution, where the early Universe experienced a faster expansion rate, which later stabilized as time progressed.
- The shear scalar is observed to be a decreasing function of cosmic time t (Figure 4), indicating a gradual reduction in anisotropy as the Universe evolves. However, the present model is not entirely shear-free, except in the special case when $m = 1$. This suggests that anisotropic effects persist throughout cosmic evolution but diminish over time, contributing to the Universe's tendency toward a more homogeneous state.
- Density is the decreasing function of cosmic time (t) (Figure 5). It approaches to zero for infinite time. The pressure varies from large negative value to small negative value (Figure 6) and it tends to zero for large time t ($t \to \infty$). The negative nature of the pressure indicates the presence of dark energy, which is responsible for driving the accelerated expansion of the Universe.
- Figure 7 shows decreasing trend i.e. it suggests the negative slope of the curvature. This behavior suggests that the model aligns with quintessence ($\omega > -1$) phase.
- Figure 8-11 shows that, All the energy conditions are observed to be a decreasing function of cosmic time in the positive region. This confirms that in our model all the standard energy conditions remain satisfied. None of the four standard energy conditions—namely the Null, Weak, Strong, and Dominant Energy Conditions—are violated.

6. Conclusion

In this research paper, we have explored a one-dimensional cosmic string coupled with a bulk viscous fluid within the framework of modified f(R, T) gravity, utilizing the Bianchi type VI_0 spacetime. To obtain exact solutions of the field equations, we imposed three specific conditions. Subsequently, we analyzed key cosmological parameters, including the jerk parameter, statefinder diagnostics, and the $O_m(z)$ parameter, to investigate the dynamic evolution of the Universe. Furthermore, we examined the energy conditions, confirming that none are violated, thereby ensuring the physical viability of our model.

The present study reveals that the cosmological model exhibits expansion, shear, and persistent anisotropy throughout its evolution. At the initial epoch (t=0), the spatial volume is zero, while as $t \to \infty$, it diverges to infinity, indicating an ever-expanding Universe. This expansion is supported by the condition $1+q > 0$. Both the expansion scalar (θ) and the

Hubble parameter (H) remain positive during cosmic evolution, gradually approaching zero as $t \to \infty$, further confirming the Universe's continuous expansion. Moreover, our analysis reveals that the model does not approach isotropy throughout the Universe's evolution, as indicated by the constant, non-zero ratio $\sigma^2/\theta^2 = \text{Constant} \neq 0$. The shear scalar, given by equation (45) and decreases over time but never vanishes entirely, further confirming the persistent anisotropy of the model. Additionally, the mean anisotropy parameter given by equation (46) remains constant for $m \neq 1$, reinforcing the anisotropic nature of the Universe. The vanishing string tension density ($\lambda=0$) suggests the absence of cosmic string effects, highlighting the dominant role of bulk viscous fluid in shaping the cosmic dynamics. The energy density (ρ) and pressure (p) are initially infinite at $t=0$ but diminish over time, approaching zero as $t \to \infty$. These results collectively highlight an expanding, anisotropic Universe with diminishing matter-energy content over cosmic time. The behavior of the jerk parameter in our model indicates a deviation from the ΛCDM model unless it approaches unity at late times. This highlights key features of cosmic acceleration, suggesting consistency with observational data and the potential presence of quintessence-like behavior. Additionally, the analysis of the $O_m(z)$parameter reveals a negative slope in its curvature, further supporting the quintessence-like nature ($\omega > -1$) of dark energy. This dynamic behavior of dark energy contributes to the late-time accelerated expansion of the Universe, aligning well with recent observational evidence. Our examination of energy conditions confirms that all standard energy conditions remain satisfied throughout the study, reinforcing the physical consistency and viability of our cosmological model. This adherence to energy conditions suggests that the model does not exhibit exotic or unphysical behavior. Furthermore, it aligns well with the fundamental principles of general relativity and observational constraints. The fulfilment of these conditions strengthens the reliability of our findings, supporting the model's applicability in describing the Universe's evolution.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

[1] A. G. Riess, A.V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks et al., Astron. J. 116, 1009 (1998). https://doi.org/10.1086/300499

[2] A G. Riess, S. Casertano, H.C. Ferguson, John Tonry, B. Mobasher, et al., Astrophys. J. 607, 665 (2004). https://doi.org/10.1086/383612

[3] S. Perlmutter, G. Aldering, G. Goldhaber, P. Nugent, S. Deustua, et al. Astrophys. J. 517, 565 (1999). https://doi.org/10.1086/307221

[4] P. Astier, J. Guy, N. Regnault, R. Pain, E. Aubourg, et al., Astron. Astrophys. 447, 31 (2006).

[5] W. M. Wood-Vasey, G. Miknaitis, C. W. Stubbs, S. Jha, A. G. Riess, et al., Astrophysics J.(2007). https://doi.org/10.48550/arXiv.astro-ph/0701041

[6] T. M. Davis, E. Mortsell, J. Sollerman, A. C. Becker, S. Blondin, et al., 666, 716 (2007). https://doi.org/10.48550/arXiv.astro-ph/00701510

[7] C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A, Kogut, et al., Astrophys. J. Suppl. 148, 1 (2003). https://doi.org/10.1086/377252

[8] D. N. Spergel, L. Verde, H.V. Peiris, E. Komatsu, M. R. Nolta et al., Astrophys. J. Suppl. 148, 175 (2003). https://doi.org/10.1086/377226

[9] K. Abazajian, S. S. Allam, N.A. Bahcall, S. F. Anderson, J. Annis et al., Astron. J. 126, 2081 (2003). https://doi.org/10.48550/arXiv.astro-ph/0305492

[10] K. Abazajian, S. S. Allam, N. A. Bahcall, S. F. Anderson, J. Annis, et al., 129, 1755 (2004). https://doi.org/10.48550/arXiv.astro-ph/0410239

[11] E. Hawkins, S. Maddox,S. Cole, O. Lahar, D. S. Madgwick, et al., Mon. Not. R. Astron. Soc. 346, 78 (2003). https://doi.org/10.1046/j.1365-2966.2003.07063.x

[12] L. Verde, A.F. Heavens, W. J. Percival, S. Matarrese, C. M. Baugh et al. Mon. Not. R. Astron. Soc. 335,432 (2002).

[13] C. Wetterich, Nucl. Phys. B 302, 668 (1988). https://doi.org/10.1016/0550-3213(88)90193-9 17. B.

[14] Ratra, J. Peebles, Phys. Rev. D 37, 321 (1988). https://doi.org/10.1103/PhysRevB.37.321

[15] T. Chiba, T. Okabe, M. Yamaguchi, Phys. Rev. D 62, ID 023511 (2000). https://doi.org/10.1103/PhysRevD.62.023511

[16] C. Armendariz-Picon, V. Mukhanov, P. J. Steinhardt, Phys. Rev. Lett. 85, 4438 (2001). https://doi.org/10.1103/PhysRevLett.85.4438

[17] R. R. Caldwell, Phys. Lett. B 545, 23 (2002). https://doi.org/10.1016/S0370-2693(02)02589-3

[18] S. Nojiri, S. D. Odinstov, Phys. Lett. B 562, 147 (2003). https://doi.org/10.1016/S0370- 2693(03)00594-X

[19] A. J. Sen, High Energy Phys. 04, ID 048 (2002). https://doi.org/10.1088/1126- 6708/2002/04/048

[20] T. Padmanabhan and T. R. Chaudhary, Phys. Rev. D 66, ID 081301 (2002). https://doi.org/10.1103/PhysRevD.66.081301

[21] E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D 70, ID 043539 (2004). https://doi.org/10.1103/PhysRevD.70.043539

[22] A. Anisimov, E. Bubichev, and A. Vikman, J. Cosmol. Astropart. Phys. 06, ID 006 (2005). https://doi.org/10.1088/1475-7516/2005/06/006

[23] D. D. Pawar, Y. S. Solanke, and S. P. Sahare, Bulg. J. Phys. 41, 60 (2014).

[24] G. C. Samanta, Int. J. Theor. Phys. 52, 3442 (2013). https://doi.org/10.1007/s10773-013-1645-9

[25] G. C. Samanta, Int. J. Theor. Phys. 52, 2303 (2013). https://doi.org/10.1007/s10773-013-1513-7

[26] D. D. Pawar and Y. S Solanke, Int. J. Theor. Phys. 53, 3052 (2014).

[27] S. Nojiri and S. D. Odinstov, Phys. Lett. D 68, ID 123512 (2003). https://doi.org/10.1103/PhysRevD.68.123512

[28] Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006). https://doi.org/10.1142/S021827180600942X

[29] T. Chiba, T. L. Smith, A. L. Erickcek, Phys. Rev. D 75, 124014 (2007). https://doi.org/10.1103/PhysRevD.75.124014

[30] T. Harko, F. S. N. Lobo, S. Nojiri, and S. D. Odintsov, Phys. Rev. D Part. Fields 84, ID 024020 (2011). https://doi.org/10.1103/PhysRevD.84.024020

[31] G. C. Samanta and R. Myrzakulov, Chin. J. Phys. 55, 1044 (2017).

[32] R. Venkateswarlu, Rao, V.U.M., Pavankumar, K.: Int. J. Thoer. Phys. 47,640-648(2008)

[33] Rao, V.U.M., Vijaya Santhi, M., Vintha, T.: Astrophys. Space sci. 314,73(2008)

[34] M. Sharif, Eur. Phys. J. Plus 133, 226 (2018). https://doi.org/10.1140/epjp/i2018-12075-1

[35] A. I. Keskin, Int. J. Mod. Phys. D 27, ID 1850112 (2018). https://doi.org/10.1142/S0218271818501122

[36] D. R. K. Reddy, R. Santikumar, and R. L. Naidu, Astrophys. Space Sci. 342, 249 (2012). https://doi.org/10.1007/s10509-012-1158-7

[37] D. R. K. Reddy, R. S. Kumar, and T. V. P. Kumar, Int. J. Theor. Phys. 52, 239 (2013). https://doi.org/10.1007/s10773-012-1325-1

[38] R. Chaubey, A.K. Shukla, Astrophys. Space Sci. 343, 415 (2013).

[39] Kibble T.W.B., Turok N., Phys. Lett. B 116, 141 (1982).

[40] Vilenkin A., Phys. Rev. D 23, 852 (1981).

[41] Letelier P.S., Phys. Rev. D 28, 2414 (1983).

[42] Goetz G., J. Math. Phys. 31, 2683 (1990).

[43] Stachel J., Phys. Rev. D 21, 2171 (1980).

[44] Kiran M., Reddy D. R. K., Astrophysics and Space Science, Vol. 346, page 521–524, (2013).

[45] Naidu R. L., et.al., Astrophysics and Space Science, Volume 348, pages 247–252, (2013).

[46] Reddy D.R.K, Shobhan Babu, P. Raju, Astrophysics and Space Science, Vol. 353, P. 275–278 (2014).

[47] Pawar D.D., Bayaskar S.N, Deshmukh A.G., Rom. Journ. Phys., Vol. 56, Nos. 5–6, P. 842–848, Bucharest, (2011).

[48] Pawar K., Dabre A. K., J. Sci. Res. 15 (3), 695-704 (2023)

[49] Bayaskar S. N., Pawar D. D., Deshmukh A. G, Rom. Journal. Phys., Vol. 54, Nos. 7–8, P. 763–770, Bucharest (2009).

[50] Pawar D.D., Bayaskar S.N., Patil V.R., Bulg. J. Phys. 36,68–75 (2009).

[51] Mete V. G., Deshmukh V. S., J. Sci. Res. 15 (2), 351-359 (2023)

[52] Bayaskar S. N., Dhanagare A.A., Inter. J. of Scientific Research in Sci. and Tech., 12(1), 174-182. (2025).

[53] Chirde V. R., Hatkar S. P., Katore S. D., Int. J. Mod. Phys. D 29, ID 2050054 (2020).

[54] Bhoyar S. R., Chirde V. R., Shekh S. H., J. Sci. Res. 11, 249 (2019).

[55] Debnath, U., & Chattopadhyay, S. International Journal of Theoretical Physics, 52, 1250-1264. (2013).

[56] Shahalam, M., Sami, S., & Agarwal, A. Monthly Notices of the Royal Astro. Society, 448(3), 2948-2959. (2015).

[57] Berman M. S., Nuovo Cimento, B, 74, 182 (1983).

[58] Collins C.B, et al. Gen. Relativ. Gravit. 12, 805 (1980).

[59] Sahni V., Shafieloo A., Starobinsky A. A., Phys. Rev. D, 78, 103502 (2008)