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(Review Article)

Deflection Analysis of straight FGM beam with different parameters

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Abstract

This paper deals with the deflection analysis of Straight functionally Graded Beam (FGM) considering various parameters. It is analysed with different boundary conditions like simply supported, clamped-clamped. The FGM beam is analysed for different L/h ratio and different volume fraction index. is analysed with different L/h ratios using higher order shear deformation theory. It was found that for a fixed value of L/h ratio as the volume fraction index is increased the maximum deflection of FGM beam also increases. Again, if the value of volume fraction index is kept constant and the L/h ratio is increased the maximum deflection also decreases. The dependence of support conditions on maximum deflection is highlighted. Comparison and convergence study has been performed to validate the present formulation. Thus, we observe that for both clamped -clamped and simply supported condition, for a fixed value of of L/h ratio as the volume fraction index is increased the maximum deflection is increased. Again, if the value of volume fraction index is kept constant and the L/h ratio is increased the maximum deflection increases.

Keywords: Straight Beam; FGM; L/H Ratio; Max Deflection; Volume Fraction

1. Introduction

Spring is very important element of an automobile as it absorbs the vehicle vibration and give the rider a comfortable ride. Leaf spring is one of those springs which are mainly used for heavy vehicles. The advantage of using a leaf spring is that it can be guided along a definite path as it deflects to acts as structural member in addition to energy absorbing device.

Lots of efforts have been made by different people to reduce the vibration of leaf spring so that the ride could be made more comfortable. In this paper the analysis of master leaf was modeled as a curved beam and vibration was studied for different materials and for different L/R ratios with different boundary conditions.

Mahmood M. et. al. [1] did the analysis of composite leaf spring using ANSYS V5.4 software and tried to optimize the weight of the spring. Abdul Rahim Abu Talib et.al [2] tried to analyse the elliptic spring made for both light and heavy trucks. They tried to optimize the spring parameters for different ellipticity ratios. Vinkel Arora et al [3] did the fatigue life assessment of leaf spring of 65Si7 using analytical and graphical methods. They also did the life assessment using SAE design manual approach and it was compared with experimental results.

Mesut Simsek et al [4] studied the vibration of straight beam within the frame work of third order shear deformation theory. They calculated the six dimensionless frequency parameters of different beams having different h/L ratios for different boundary conditions.

Ankit Gupta and Mohammad Talha [5] introduced the geometrically nonlinear vibrations response of FGM plates. They proposed displacement based new hyperbolic higher-order shear and normal deformation theory (HHSNDT). The

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performance of a curved beam with coupled polynomial distributions was investigated by P. Raveendranath et al [6]. The result shows excellent convergence of natural frequencies even for very thin deep arches and higher vibrational modes.

- M. Kawakami [7] presented an approximate method to study the analysis for both the in-plane and out-of-plane free vibration of horizontally curved beams with arbitrary shapes and variable cross-sections.
- S.M. Ibrahim et al [8] investigated large amplitude periodic forced vibration of curved beams under periodic excitation using a three-noded beam element.
- Mohammad Amir and Mohammad Talha [9] analyzed the thermo-elastic vibration of shear deformable functionally graded material (FGM) curved beams with micro structural defects by the finite element method. The material properties of FGM beams are allowed to vary continuously in the thickness direction by a simple
- power-law distribution in terms of the volume fractions of the constituents. Mohammad Amir and Mohammad Talha [10] studied the imperfection sensitivity in the vibration behavior of functionally graded arches with micro structural defects (porosity).
- Vaibhav Ghodge [11] did the modal analysis of a cantilever beam and simply supported beam using ANSYS for Aluminum alloy, grey cast iron, Structural steel and Copper alloy.

Yiming Fu [12] studied nonlinear free vibration and dynamic stability for the piezoelectric functionally graded beams, subjected to one-dimensional steady heat conduction in the thickness direction. J. Yuan and S. M. Dickinson [13] used Rayleigh-Ritz method for the solution of the free vibration problem of systems comprised of straight and/or curved beam components. Eigen value problems of Timoshenko and shear-deformable curved beams were analyzed by S.Y. Yang andH.C. Sin [14] using the elements with six degrees of freedom. The results of the Eigen-analysis show that the curvature-based beam elements are free of locking and are efficient. Mehdi Hajianmaleki and Mohammad S. Qatu [15] on the basis of exhaustive study on the vibration analysis of composite thin and thick beams explained a simple classic and shear deformation model that can be used for beams with any laminate. P. Chidambaram and A.W. Leissa [16] did a study on the Vibrations of planar curved beams, rings, and arches and special attention was given to the effects of initial static loading, nonlinear vibrations and the application of finite element techniques.

2. Mathematical Formulation

2.1. Displacement field

$$u(x,z,t) = u_0(x,t) + z\emptyset_x(x,t) - \frac{4z^3}{h^2}\left[\emptyset_x(x,t) + \frac{\partial w_0}{\partial x}(x,t)\right]$$

$$w(x,z,t) = w_0(x,t) \qquad (3.1)$$

$$let\ \frac{\partial w_0}{\partial x} = \theta_x$$

$$u(x,z,t) = u_0(x,t) + z\emptyset_x(x,t) - \frac{4z^3}{3*h^2}[\emptyset_x(x,t) + \theta_x(x,t)]$$

$$u(x,z,t) = u_0(x,t) + \emptyset_x\left(z - \frac{4z^3}{3h^2}\right) - \frac{4z^3}{3h^2}\theta_x(x,t)$$

$$w(x,z,t) = w_0(x,t)$$

Where, u and w represents the displacements of a point along the (x,z) coordinates. u_0, $w_{0,}\emptyset_x$, θ_x are four unknown displacement functions of mid-plane.

$$\begin{bmatrix} u(x,z,t) \\ w(x,z,t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \left(z - \frac{4z^3}{3h^2}\right) & -\frac{4z^3}{3h^2} \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ w_0 \\ \emptyset_x \\ \theta_x \end{bmatrix} \qquad (3.2)$$

The basic field variables can be symbolized mathematically as, { λo } = $\{u_0, w_{0,}\emptyset_x , \theta_x\}^T$

$$\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \left(z - \frac{4z^3}{3h^2}\right) & -\frac{4z^3}{3h^2} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} [\boldsymbol{\lambda} o]$$

The displacement field in the compressed form can be written as

$$[\mathrm{U}] = [\check{\mathrm{N}}]\ [\boldsymbol{\lambda} o]$$

$$\text{Where, } [\mathrm{U}] = \begin{bmatrix} u \\ w \end{bmatrix} \textbf{ and } [\check{\mathrm{N}}] = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \left(z - \frac{4z^3}{3h^2}\right) & -\frac{4z^3}{3h^2} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

2.2. Strain-displacement relations

Strain displacement relation is written below for curved beam and for the straight beam the value of R is taken as infinity.

$$\varepsilon_{xx} = 1/(1 + \frac{z}{R})(\frac{\partial u}{\partial x} + \frac{w}{R})$$

Using linear strain displacement relations, the strain vectors may be defined as

$$\varepsilon_{xx} = 1/(1 + \frac{z}{R})\{\frac{\partial u_0}{\partial x} + z\frac{\partial \emptyset_x}{\partial x} - \frac{4z^3}{h^2}\left[\frac{\partial \emptyset_x}{\partial x} + \frac{\partial \theta_x}{\partial x}\right] + \frac{w_0}{R}\} \qquad (3.3)$$

$$\Upsilon_{xz} = \frac{\partial u}{\partial z} + 1/(1 + \frac{z}{R})(\frac{\partial w}{\partial x} - \frac{u}{R}) \qquad (3.4)$$

$$= (\frac{\partial u_0}{\partial z} + \emptyset_x - \frac{4z^2}{h^2}[\emptyset_x + \theta_x]) + 1/(1 + \frac{z}{R})\{\frac{\partial w_0}{\partial x} - \frac{u_0}{R} - \frac{z}{R}(\emptyset_x) - \frac{4z^3}{3h^2}[\emptyset_x + \theta_x]\}$$

As u_0 *is a function of x and t only . ie* $\frac{\partial u_0}{\partial z} = 0.$

$$\Upsilon_{xz} = 1/(1 + \frac{z}{R})\{(1 + \frac{z}{R})(\emptyset_x - \frac{4z^2}{h^2}[\emptyset_x + \theta_x]) + \frac{\partial w_0}{\partial x} - \frac{u_0}{R} - \frac{z}{R}(\emptyset_x) - \frac{4z^3}{3 * R * h^2}[\emptyset_x + \theta_x]$$

$$\Upsilon_{xz} = 1/(1 + \frac{z}{R})\{(\emptyset_x + \frac{\partial w_0}{\partial x} - \frac{u_0}{R}) - \frac{4z^2}{h^2}[\emptyset_x + \theta_x] - \frac{8z^3}{3 * R * h^2}[\emptyset_x + \theta_x]\}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \Upsilon_{xz} \end{bmatrix} = 1/(1 + \frac{z}{R})\left(\begin{bmatrix} \frac{\partial u_0}{\partial x} + \frac{w_0}{R} \\ \emptyset_x + \frac{\partial w_0}{\partial x} - \frac{u_0}{R} \end{bmatrix} + z\begin{bmatrix} \frac{\partial \emptyset_x}{\partial x} \\ 0 \end{bmatrix} + z^2\begin{bmatrix} 0 \\ -\frac{4}{h^2}[\emptyset_x + \theta_x] \end{bmatrix} + z^3\begin{bmatrix} -\frac{4}{3h^2}\left(\frac{\partial \emptyset_x}{\partial x} + \frac{\partial \theta_x}{\partial x}\right) \\ -\frac{8}{3R * h^2}[\emptyset_x + \theta_x] \end{bmatrix} \right)$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \Upsilon_{xz} \end{bmatrix} = (1/(1 + \frac{z}{R}))(\begin{bmatrix} \epsilon^\circ{}_1 \\ \epsilon^\circ{}_2 \end{bmatrix} + z\begin{bmatrix} k_1^1 \\ 0 \end{bmatrix} + z^2\begin{bmatrix} 0 \\ k_2^2 \end{bmatrix} + z^3\begin{bmatrix} k_1^3 \\ k_2^3 \end{bmatrix})$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \Upsilon_{xz} \end{bmatrix} = 1/(1 + \frac{z}{R})\begin{bmatrix} \mathbf{1} & \mathbf{0} & \boldsymbol{z} & \mathbf{0} & \mathbf{0} & z^3 & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \boldsymbol{z} & z^2 & \mathbf{0} & z^3 \end{bmatrix}\begin{bmatrix} \epsilon^\circ{}_1 \\ \epsilon^\circ{}_2 \\ k_1^1 \\ k_2^2 \\ k_1^3 \\ k_2^3 \end{bmatrix}$$

$$[\mathrm{T}] = 1/(1 + \frac{z}{R})\begin{bmatrix} \mathbf{1} & \mathbf{0} & \boldsymbol{z} & \mathbf{0} & \mathbf{0} & z^3 & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \boldsymbol{z} & z^2 & \mathbf{0} & z^3 \end{bmatrix}$$

$$where, \qquad \epsilon^\circ{}_1 = \frac{\partial u_0}{\partial x} + w_0/R$$

$$\epsilon^{\circ}{}_2 \;=\; \emptyset_x + \frac{\partial w_0}{\partial x} - \frac{u_0}{R}$$

$$k_1^1 = \frac{\partial \emptyset_x}{\partial x}$$

$$k_2^2 = \; -\frac{4}{h^2}[\emptyset_x + \theta_x]$$

$$k_1^3 = \; -\frac{4}{3h^2}\left(\frac{\partial \emptyset_x}{\partial x} + \frac{\partial \theta_x}{\partial x}\right)$$

$$k_2^3 = \; -\frac{8}{3R * h^2}[\emptyset_x + \theta_x]$$

$$\begin{bmatrix} \epsilon^{\circ}{}_1 \\ \epsilon^{\circ}{}_2 \\ k_1^1 \\ k_2^2 \\ k_1^3 \\ k_2^3 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\mathbf{1}}{\boldsymbol{R}} & \mathbf{0} & \mathbf{0} \\ -\frac{\mathbf{1}}{\boldsymbol{R}} & \frac{\partial}{\partial x} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{\partial}{\partial x} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{4}{h^2} & -\frac{4}{h^2} \\ \mathbf{0} & \mathbf{0} & \frac{-4}{3h^2} & \frac{-4}{3h^2} \\ \mathbf{0} & \mathbf{0} & \frac{-8}{3 * R * h^2} & \frac{-8}{3 * R * h^2} \end{bmatrix}$$

The linear constitutive relations of an isotropic beam are given as

$$\begin{Bmatrix} \boldsymbol{\sigma_{xx}} \\ \boldsymbol{\tau_{xz}} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{Q_{11}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q_{44}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon_{xx}} \\ \boldsymbol{\gamma_{xz}} \end{Bmatrix}$$

Where Q_{11}=E/(1-mu^2) and Q_{44} =E/(2*(1+mu)).

$$\begin{Bmatrix} \boldsymbol{\sigma_{xx}} \\ \boldsymbol{\tau_{xz}} \end{Bmatrix} = \boldsymbol{QT\varepsilon}^{\circ}$$

2.3. Finite element formulation

A two noded element with four degrees of freedom per node is being utilized to discretize the beam geometry. The generalized displacement vector of the model is expressed by

{Λ}=[N]{Λ^e } and {ϵ}=[B]{Ei } where {Λ^e} is the nodal displacement vector, [N] is the shape function matrix and [B] is the strain displacement matrix. N1 and N2 are Lagrangian interpolation function.

The strain energy of the curved beam is given by

$$U = \frac{1}{2}\int \{\varepsilon\}^T\{\sigma\}dV = \frac{1}{2}\int \{\varepsilon\}^T[Q_i]\{\varepsilon\}dV$$

$$= \frac{1}{2}\int \{\varepsilon^0\}^T[T_{tr}]^T\,[Q_i]\,[T_{tr}]\,\{\varepsilon^0\}\,dV = \frac{1}{2}\int \{\Lambda^e\}^T\,[B]^T[D][B]\{\Lambda^e\}dA$$

$$= \frac{1}{2}\{\Lambda^e\}^T[K^e]\{\Lambda^e\}$$

The kinetic energy of the FGM curved beam is given by

$$T_k = \frac{1}{2}\int \rho\{\dot{U}\}^T\{\dot{U}\}dV = \frac{1}{2}\int \rho\{\dot{\Lambda}^e\}^T[\bar{N}]^T\,[\bar{N}]\{\dot{\Lambda}^e\}dV$$

$$= \frac{1}{2}\int_A \{\dot{\Lambda}^e\}^T[m]\,\{\dot{\Lambda}^e\}dA = \frac{1}{2}\int_A \{\dot{\Lambda}^e\}^T[N]^T[m]\,[N]\{\dot{\Lambda}^e\}dA$$

$$= \frac{1}{2}\{\dot{\Lambda}^e\}^T[M^e]\{\dot{\Lambda}^e\}$$

2.3.1. Governing Equation

Using variational principal, the equation for frequency is derived

$$\frac{\partial E_T}{\partial x} = 0 = \int \boldsymbol{\varepsilon^T \sigma dv} + \int \boldsymbol{\rho U^T\, U dV}$$

$$\mathbf{0} = \int \boldsymbol{\varepsilon^{\circ T}\, T^T Q T \varepsilon^\circ dv} + \int \boldsymbol{\rho\, \lambda_0^T N^T N \lambda_0 dV}$$

$$= \boldsymbol{b}\int_0^L \int_{-h/2}^{h/2} \boldsymbol{\varepsilon^{\circ T} T^T Q T \varepsilon^\circ dz\, dx} + \boldsymbol{b}\int_0^L \int_{-h/2}^{h/2} \boldsymbol{\rho \lambda_0^T N^T N \lambda_0 dz\, dx}$$

$$= \boldsymbol{b}\int_0^L \boldsymbol{\varepsilon^{\circ T}}[\boldsymbol{D}]\boldsymbol{\varepsilon^\circ\, dx} + \mathrm{b}\int_0^L \boldsymbol{\lambda_0^T}[\,\boldsymbol{m}]\boldsymbol{\lambda_0 dx}$$

Where, D= $\int_{-h/2}^{h/2} \boldsymbol{T^T Q T dz}$ and m= $\int_{-h/2}^{h/2} \boldsymbol{N^T N dz}$

$$= \mathrm{b}\int_0^L \boldsymbol{\lambda_0^T B^T\, D B \lambda_0 dx} + \mathrm{b}\int_0^L \boldsymbol{\lambda_0^T}[\,\boldsymbol{m}]\boldsymbol{\lambda_0 dx}\;\; \boldsymbol{X}$$

$$= \mathrm{K\,U} + \mathrm{M}\boldsymbol{\ddot{U}}$$

$$\mathrm{KU} = \boldsymbol{\lambda}\ \mathrm{MU,\ where}\ \boldsymbol{\lambda} = \boldsymbol{\omega^2}$$

3. Result and Discussion

3.1. Deflection analysis of beam with different parameters

The deflection analysis of straight isotropic beam is done with the mentioned load and it is found as the L/h ratio is increase the value of deflection increases for all the support condition. It is found that the deflection of C-F beam increasing is as presented in figure 4.4. The least deformation is observed in C-C beam for any L/h ratio.As far as the volume fraction is concerned it is found that the deflection of the beam increases with increase in volume fraction index for the combination of material mentioned in table 4.1 for different boundary condition of the Beam.

Table 1 Maximum Deflection with variation in Volume Fraction Index

L/h ratio	Max. deflection with diff. volume fraction index, C-C Beam			
	0.5	2	5	10
5	0.008927	0.01499	0.0196	0.02247
20	0.4159	0.6942	0.8288	0.9139
50	6.349	10.58	12.55	13.78
100	50.44	84.04	99.54	109.4

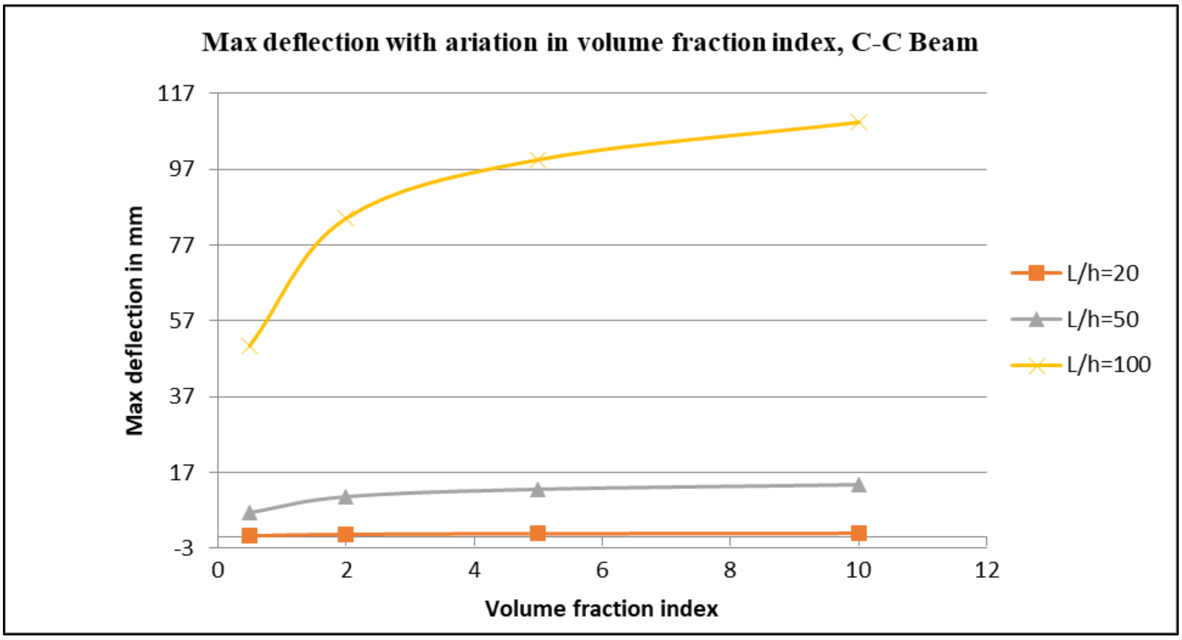

Figure 1 Maximum Deflection with variation in Volume Fraction Index

Table 2 Maximum Deflection with variation in L/h ratio

Volume fraction index	Max. deflection with variation in L/h ratio, C-C Beam			
	5	20	50	100
0.5	0.008927	0.4159	6.349	50.44
2	0.01499	0.6942	10.58	84.04
5	0.0196	0.8288	12.55	99.54
10	0.02247	0.9139	13.78	109.4

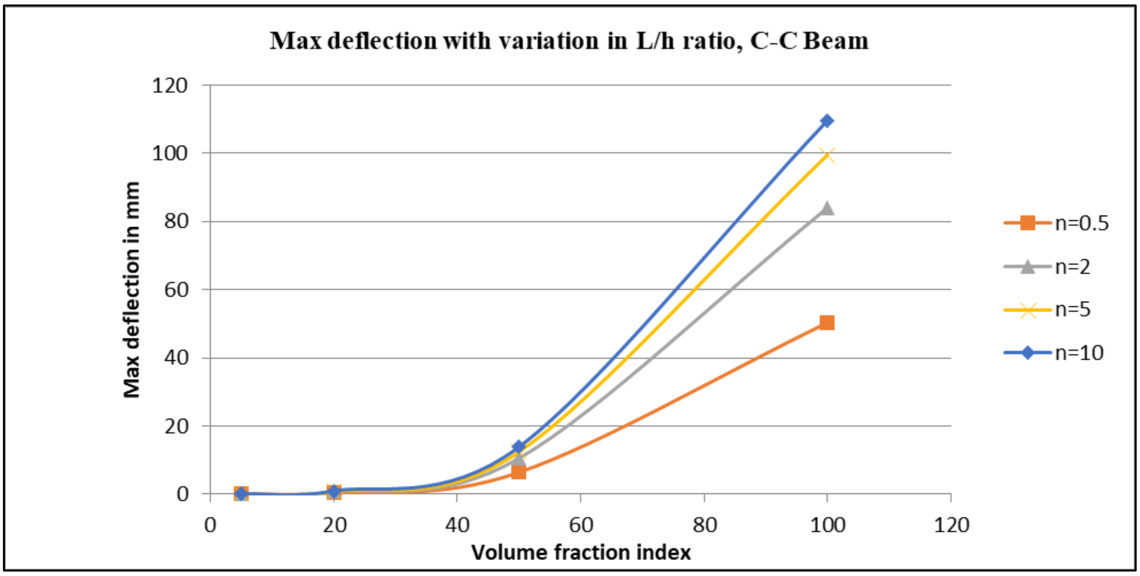

Figure 2 Maximum Deflection with variation in L/h ratio C-C Beam

Table 3 Maximum Deflection with variation Volume Fraction Index

L/h ratio	Max. deflection with diff. volume fraction index, S-S Beam			
	0.5	2	5	10
5	0.02664	0.03922	0.04928	0.05844
20	1.548	2.24	2.72	3.208
50	24	34.68	42.04	49.57
100	190.1	274.9	333.5	393.4

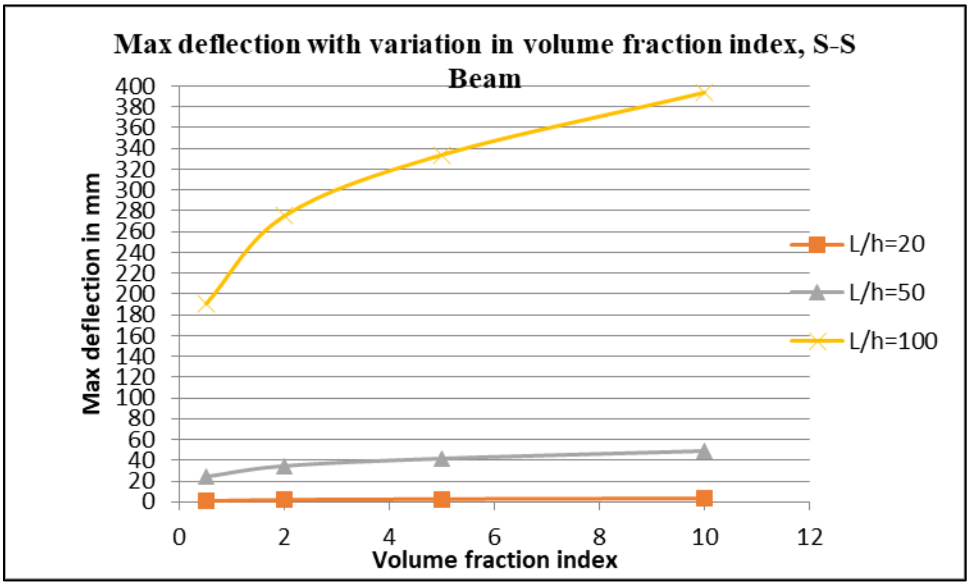

Figure 3 Maximum Deflection with variation Volume Fraction Index

Table 4 Maximum Deflection with variation in L/h Ratio, S-S Beam

Volume fraction index	Max. deflection with diff. L/h ratio, S-S Beam			
	5	20	50	100
0.5	0.02664	1.548	24	190.1
2	0.03922	2.24	34.68	274.9
5	0.04928	2.72	42.04	333.5
10	0.05844	3.208	49.57	393.4

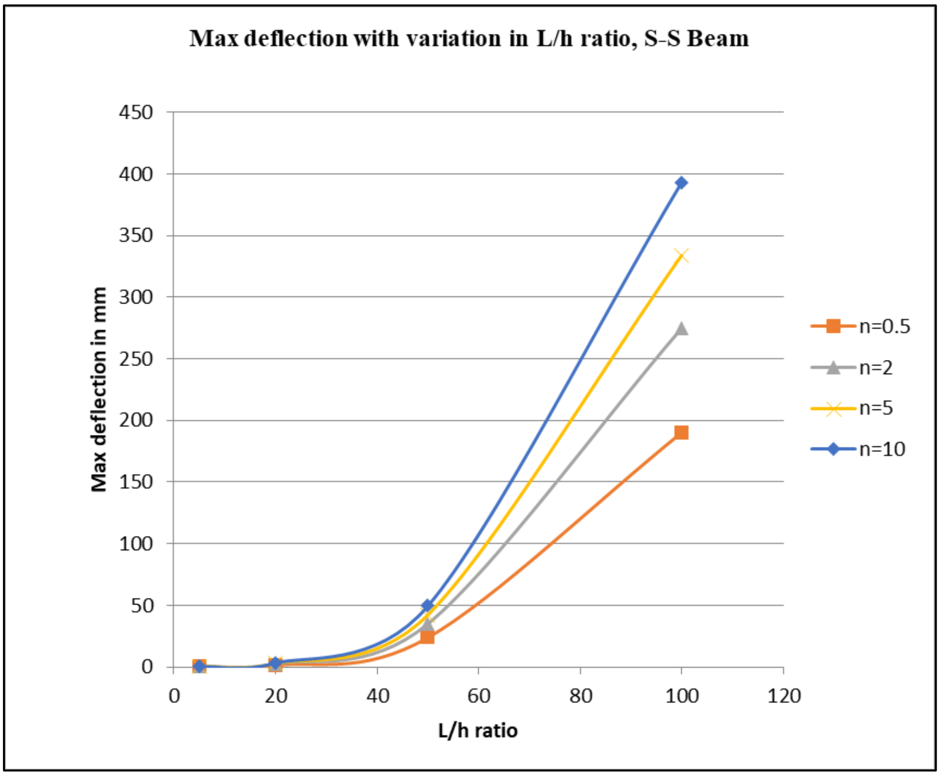

Figure 4 Maximum Deflection with variation in L/h Ratio, S-S Beam

4. Conclusion

The deflection analysis straight FGM was done under various conditions. MATLAB was used for the analysis. The effect of end condition on the vibration of straight FGM beam was studied for L/ h =5,20,50 and 100. The effect of change in volume fraction index and support condition is also studied. The volume fraction index was taken as 0.5,2,5 and 10. The result was also validated. During the analysis it was found that as the L/h ratio is increased the value of maximum deflection in both the clamped-clamped condition and simply supported condition increases. We also tried to observe the effect on maximum deflection on volume fraction index and it was found that it increases with increase in volume fraction index for both clamped and simply supported condition.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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