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Bridging theoretical and applied gaps in the analysis and control of neutral fractional-order systems using advanced controllability and numerical analysis methodologies

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Abstract

The objective of this article is to create a comprehensive mathematical formula for the analysis and control of neutral fractional-order systems that are characterized by state-dependent delays and periodic coefficients. Some of the issues that this article will address involve a variety of important area, including, but not limited to, the existence, uniqueness, stability and controllability of neutral fractional-order systems and will develop efficient numerical methods. This study will also integrate recent developments in the fractional calculus with the application of develop sufficient conditions for asymptotic stability criteria for neutral fractional-order systems using the fractional Lyapunov-Krasovskii functional and an Algorithm of Linear Matrix Inequalities. The use of a predictor-corrector method with adaptive step size control and cubic spline interpolation with state-dependent delays will further aid in providing a theoretical basis for the development of adequate control of neutral fractional-order systems. Extensive numerical simulations and theoretical validation of the methods discussed in this paper have shown substantial improvements in control accuracy and computational efficiency when applied to biomedical engineering and smart grids.

Keywords: Neutral Fractional-Order Systems; State-Dependent Delay; Period Coefficients; Stable Controllability; Asymptotic Stability; Predictor-Corrector Methods; Applications in Biomedical Fields

1. Introduction

Fractional order systems have changed how complicated dynamic systems that exhibit memory and hereditary attributes can be modeled and analyzed [1]. A key category of the fractional order systems is the neutral type, which is characterized by the fact that the derivatives of the 'effect' of a past input can be obtained from the current state variable as well as the delayed state variables. Examples of neutral fractional order systems exist in various fields from biology to engineering [2]. The inclusion of state dependent delays and periodic coefficients in the modelling of neutral order systems introduces many mathematical challenges that have not been adequately addressed in existing literature [3].There exists a research gap with regard to the effect of combining state dependent delays, periodic coefficients, and neutral fractional order systems due to the inadequacies of available research and resources examining the link between the theoretical analysis and the actual application of the models. The contribution of this paper is to outline the rigorous existence and uniqueness of solutions under generalized conditions, to develop new criteria for stability via fractional Lyapunov-Krasovskii theory and to ensure that the controlled behavior of the neutral fractional order systems can be characterized precisely using advanced Gramian theory [4,5]. This research will also address the efficient implementation of numerical solutions for these complex systems, utilizing numerical methods with guaranteed convergence properties, by providing examples from biological and engineering applications.

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2. Preliminaries

2.1. Fractional calculus foundations

Definition 2.1 (Caputo fractional derivative [6,7]): for n-1< α < n

$$^{c}D_{t}^{\alpha} = \frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)\,d\tau$$

Definition 2.2 (Riemann-Liouville integral [8]):

$$I_{t}^{\alpha}\,f(t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}\,f(\tau)\,d\tau$$

2.2 System formulation

The neutral fractional-order system with state-dependent delay [9,10]:

$$D_{t}^{\alpha}\left[x(t) - N\left(t, x\left(t-\tau\big(x(t)\big)\right)\right)\right] = A(t)x(t) + B(t)u(t) + G(t)x\left(t-\tau\big(x(t)\big)\right)$$

Where: state vector $x(t) \in R^{n}$, control input $u(t) \in R^{m}$

$A(t), B(t), G(t)$ Periodic matrices with period T, $\tau(x(t))$ state-depenedent delay, $0 \leq \tau(x(t)) \leq \tau_{max}$, $N(\cdot)$ neutral term for function

Initial condition $x(t) = \emptyset(t)$ for $t \in [-\tau_{max}, 0]$

3. Theoretical framework with proofs

3.1. Existence and uniqueness

Theorem 3.1 (Existence and uniqueness)

Consider the neutral fractional-order system with state-dependent delay and periodic coefficients [8-12].

$$^{c}D_{t}^{\alpha}\left[x(t) - N\left(t, x\left(t-\tau\big(x(t)\big)\right)\right)\right] = A(t)x(t) + B(t)u(t) + G(t)x\left(t-\tau\big(x(t)\big)\right),$$

With initial condition x (t) = $\varphi(t)$ for t∈ $[-t_{max}, 0]$

Assume the following conditions hold

Lipchitz conditions :

$$\|N(t,x_2) - N(t,x_1\| \leq L_N\|x_2 - x_1\|\ \forall x_1, x_2 \in R^{n}\ , \forall t \in [0,T]$$

$$\|G(t,x_2) - G(t,x_1\| \leq L_G\|x_2 - x_1\|$$

$$\|\tau(x_2) - \tau(x_1\| \leq L_\tau\|x_2 - x_1\|$$

Boundedness :$\tau_{max} \leq \tau(x) \leq 0$

The functions A(t), B(t), G(t) are continuous and periodic T.

Contraction condition :$L_N \leq 1$

Then the system has unique solution on the interval [0, T] for any continuous initial condition $\varphi(t)$

Proof:

Step1: Reformulate as integral equation using the properties of caputo derivative and the initial condition, we rewrite the system in equivalent form:

$$x(t) - N\left(t, x\left(t - \tau\left(x(t)\right)\right)\right) = \varphi(0) - N\left(0, \varphi\left(-\tau\left(\varphi(0)\right)\right)\right) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[A(s)x(s) + B(s)u(s) + G(s)x\left(s - \tau\left(x(s)\right)\right)\right] ds$$

Define the operator Φon the space of continuous function c $([-\tau_{max}, T], R^n)$ by:

$$(\Phi x)(t) = \begin{cases} \varphi(t) \ \ for\ t \in [-\tau_{max}, 0] \\ \phi(0) - N\left(0, \varphi\left(-\tau\left(\varphi(0)\right)\right)\right) + N\left(t, x\left(t - \tau\left(x(t)\right)\right)\right) + \\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[A(s)x(s) + B(s)u(s) + G(s)x(s - \tau(x(s)))]ds \end{cases} \quad \text{for } t \in [0, t]$$

Step 2: Φ is a contraction mapping

Let x, y ∈ C$([-\tau_{max}, T], R^n)$. For t ∈ [0, T] consider:

$$\begin{aligned} \|(\Phi x)(t) - (\Phi y)(t)\| &\leq \left\|N\left(t, x\left(t - \tau\left(x(t)\right)\right)\right) - N\left(t, y\left(t - \tau\left(y(t)\right)\right)\right)\right\| \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|A(s)[x(s) - y(s)]\| ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|G(s)[x\left(s - \tau\left(x(s)\right)\right) - y\left(s - \tau\left(y(s)\right)\right)]\right\| ds \end{aligned}$$

Using the Lipschitz conditions:

$$\begin{aligned} \|(\Phi x)(t) - (\Phi y)(t)\| &\leq L_N\left\|x\left(t - \tau\left(x(t)\right)\right) - y\left(t - \tau\left(y(t)\right)\right)\right\| + \frac{M_A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s) - y(s)\| ds \\ &+ \frac{M_G}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|x\left(s - \tau\left(x(s)\right)\right) - y(s - \tau\left(y(s)\right))\right\| ds \end{aligned}$$

Where $M_A = sup\|A(t)\|_{t\in[0,T]}$, $M_G = sup\|G(t)\|_{t\in[0,T]}$

Step 3: Estimate the delay terms

For the delay terms, we use the triangle inequality and Lipschitz conditions:

$$\left\|x\left(t - \tau\left(x(t)\right)\right) - y\left(t - \tau\left(y(t)\right)\right)\right\| \leq \left\|x\left(t - \tau\left(x(t)\right)\right) - y\left(t - \tau\left(x(t)\right)\right)\right\| + \left\|y\left(t - \tau\left(x(t)\right)\right) - y\left(t - \tau\left(y(t)\right)\right)\right\|$$

$$\leq \|x - y\|_\infty + L_y L_\tau \|x - y\|_\infty$$

$$= (1 + L_y L_\tau)\|x - y\|_\infty$$

Where L_y is the Lipschitz constant of y (which exists since y continuous on compact set)

Step 4: combine estimates

Substituting back

$$\|(\Phi x)(t) - (\Phi y)(t)\| \leq L_N(1 + L_y L_\tau)\|x - y\|_\infty + \frac{M_A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, ds\|x - y\|_\infty + \frac{M_G}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1 + L_y L_\tau)\|x - y\|_\infty ds$$

Evaluating the integrals:

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha} \leq \frac{T^\alpha}{\alpha}$$

Thus:

$$\|(\Phi x)(t) - (\Phi y)(t)\| \leq [L_N(1 + L_y L_\tau) + \frac{M_A T^\alpha}{\Gamma(\alpha+1)} + \frac{M_G T^\alpha (1 + L_y L_\tau)}{\Gamma(\alpha+1)}]\|x - y\|_\infty$$

Step 5: verify contraction condition

Let $K = L_N(1 + L_y L_\tau) + \frac{M_A T^\alpha}{\Gamma(\alpha+1)} + \frac{M_G T^\alpha}{\Gamma(\alpha+1)}$

Since $L_N < 1$ we can choose t sufficiently small such that $K < 1$ specifically

$$T < [\frac{(1 - L_N)(1 + L_y L_\tau)\Gamma(\alpha+1)}{M_A + M_G(1 + L_y L_\tau)}]^{1/\alpha}$$

Then Φ is contraction on $C([-\tau_{max}, T\,], R^n)$

Step 6: Apply banach fixed-point theorem

By the banach fixed point theorem since Φ is contraction on the complete metric space $C([-\tau_{max}, T\,], R^n)$ there exists unique fixed point x^* such that

$\Phi x^* = x^*$

This fixed point is the unique solution of the original system on $[0, T]$

Step 7: Extend to arbitrary time interval the time T obtained in step 5 depends only on the Lipschitz constant and system parameters, not on the initial condition. Therefor we can repeat the process on intervals [T,2T],[2T,3T],..... to extend the solution to any finite time interval.

Theorem 3.2 (asymptotic stability via fractional Lyapunov-Krasovskii method) Consider the neutral fractional-order system with state-dependent delay:

$${}^{c}D_t^\alpha \left[x(t) - N\left(t, x\left(t - \tau(x(t))\right)\right)\right] = A(t)x(t) + B(t)u(t) + G(t)x\left(t - \tau(x(t))\right),$$

With u (t) = 0 for stability analysis

Assume there exists a Lyapunov-Krasovskii fractional v (t,x_t) of the form :

$$V(t, x_t) = \left[x(t) - N\left(t, x\left(t - \tau(x(t))\right)\right)\right]^T P[x(t) - N\left(t, x(t - \tau(x(t))\right))] + \int_{t-\tau_{max}}^t x(s)^T Q x(s) d + \int_{t-\tau(x(t))}^t \left({}^{c}D_S^\beta(s)\right)^T R\, {}^{c}\dot{D}_S^\beta(s) ds \text{ when } 0 < \beta \leq 1$$

Where $P, Q, R \in R^{n\times n}$ are symmetric positive define the matrices.

If Caputo fractional derivative of V along the system trajectories satisfies

$${}^{c}D_t^{\alpha}V(t,x_t) \le -\epsilon\|x(t)\|^2$$

for some $\epsilon > 0$ then the zero equilibrium of the system asymptotically stable

Proof:

First we show that v $V(t,x_t)$ positive definite and decrescent.

Lower bound

$$\mathrm{V}(\mathrm{t},x_t) \ge \lambda_{min}(P)\left\|x(t) - N\left(t, x\left(t-\tau\big(x(t)\big)\right)\right)\right\|^2 + \lambda_{min}(Q)\int_{t-\tau_{max}}^{t}\|x(s)\|^2\,ds + \lambda_{min}(R)\int_{t-\tau(x(t))}^{t}\left\|{}^{c}\dot{D}_s^{\beta}(s)\right\|^2 ds$$

Using the Lipschitz condition on N *with* $L_N < 1$:

$$\left\|x(t) - N\left(t, x\left(t-\tau\big(x(t)\big)\right)\right)\right\| \ge \|x(t)\| - \left\|N\left(t, x\left(t-\tau\big(x(t)\big)\right)\right)\right\| \ge (1-L_N)\|x(t)\|$$

Thus:

V $(\mathrm{t},x_t) \ge c_1\|x(t)\|^2 + c_2\int_{t-\tau_{max}}^{t}\|x(s)\|^2 ds$ where $c_1 = \lambda_{min}(P)(1-L_N)^2 > 0$ $c_2 = \lambda_{min}(Q) > 0$

Upper bound

$$\mathrm{V}(\mathrm{t},x_t) \le \lambda_{max}(P)\left\|x(t) - N\left(t, x\left(t-\tau\big(x(t)\big)\right)\right)\right\|^2 + \lambda_{max}(Q)_{\tau_{max}} sup_{s\in[t-\tau_{max},t]}\|x(s)\|^2 + \lambda_{max}(R)_{\tau_{max}} sup_{s\in[t-\tau_{max},t]}\left\|{}^{c}D_S^{\beta}(s)\right\|^2$$

Using the Lipschitz condition:

$$\left\|x(t) - N\left(t, x\left(t-\tau\big(x(t)\big)\right)\right)\right\| \le (1+L_N)\|x(t)\|$$

Therefore:

$V(t,x_t) \le c_3\|x(t)\|^2 + c_4 sup_{s\in[t-\tau_{max},t]}\|x(s)\|^2$ Where $c_3, c_4 > 0$

Hence V is positive definite and decrescent

Step 2: Analyze the fractional derivative of V

Using the Leibniz rule for fractional derivative and system dynamics:

$${}^{c}D_t^{\alpha}V(t,x_t) = 2[x(t) - N\left(t, x\left(t-\tau\big(x(t)\big)\right)\right)]^T P\,{}^{c}D_t^{\alpha}\left[x(t) - N\left(t, x\left(t-\tau\big(x(t)\big)\right)\right)\right] + {}^{c}D_t^{\alpha}\left[\int_{t-\tau(x(t))}^{t} x(s)^T Q x(s) ds\right] + {}^{c}D_t^{\alpha}\left[\int_{t-\tau(x(t))}^{t} x(s)^T R\,{}^{c}\dot{D}_s^{\beta}(s) ds\right] + remainder\ terms$$

Substitute the system dynamics:

$${}^{c}D_t^{\alpha}\left[x(t) - N\left(\left(t, x\big(t-\tau(x(t))\big)\right)\right)\right] = A(t)x(t) + G(t)x\big(t-\tau\big(x(t)\big)\big)$$

Thus:

$$\square^c D_t^\alpha V(t,x_t) = 2[x(t) - N\left(t, x\left(t-\tau\big(x(t)\big)\right)\right)]^T P\left[A(t)x(t) + G(t)x\left(t-\tau\big(x(t)\big)\right)\right] + \square^c D_t^\alpha[\int_{t-\tau_{max}}^{t} x(s)^T Qx(s)ds] + \square^c D_t^\alpha[\int_{t-\tau(x(t))}^{t} \square^c \dot{D}_s^\beta(s)^T R \square^c D_s^\beta(s)ds$$

Step 3: bound the integral terms

Using fractional differentiation under the integral sign:

$$\square^c D_t^\alpha[\int_{t-\tau_{max}}^{t} x(s)^T Qx(s)ds] \le x(t)^T Qx(t) - x(t-\tau_{max})^T Qx(t-\tau_{max}) + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{\int_{t-\tau_{max}}^{t} x(s)^T Qx(s)ds}{(t-\xi)^{\alpha+1}}\, d\xi$$

Similarly for the second integral:

$$\square^c D_t^\alpha\left[\int_{t-\tau(x(t))}^{t} \square^c \dot{D}_s^\beta(s)^T R \square^c D_s^\beta(s)ds\right] \le \square^c \dot{D}_s^\beta(t)^T R \square^c \dot{D}_s^\beta(t) - \left[1-\dot{\tau}\big(x(t)\big)\right]\square^c \dot{D}_s^\beta\left(t-\tau\big(x(t)\big)\right)^T R \square^c D_s^\beta\left(t-\tau\big(x(t)\big)\right) + remainder$$

Step 4: derive the liner matrix inequality (LIM) condition

Collect all terms and express in quadratic form. Define the augmented state vector

$\eta(t) = \begin{bmatrix} x(t) \\ x(t-\tau\big(x(t)\big)) \\ \dot{x}(t-\tau\big(x(t)\big)) \end{bmatrix}$ After detailed calculations, we obtain:

$V(t,x_t) \le \eta(t)^T \Theta(t)\eta(t) +$ higher order terms

Where $\theta(t)$ is a symmetric matrix depending on P, Q, R, A (t), G (t),L_N, L_τ .

The sufficient condition for asymptotic stability is:

$\theta(t) + \varepsilon I \le 0$ for all $t \ge 0$

This leads to the linear matrix inequality (LIM):

$$\begin{bmatrix} \Theta_{11}(t) & \Theta_{12}(t) & \Theta_{13}(t) \\ \Theta_{12}^T(t) & \Theta_{22}(t) & \Theta_{23}(t) \\ \Theta_{13}^T(t) & \Theta_{23}^T(t) & \Theta_{33}(t) \end{bmatrix} \le -\varepsilon I$$

Where the matrix blocks are:

$$\Theta_{11} = A(t)^T P + PA(t) + Q + terms\ from\ neutral\ and\ delay\ parts$$

$\Theta_{12} = PG(t) - A(t)^T PN(\cdot) + cross\ terms$

$\Theta_{13} = terms\ involving\ \square^c D_s^\beta(t-\tau\big(x(t)\big))$

Step 5: apply fractional Lyapunov stability theory

By the fractional Lyapunov direct method [10-14], if:

V(t,x_t) is positive and decrescent

$$^{c}D_t^{\alpha}V(t,x_t) \le -\varepsilon\|x(t)\|^2 \; for \; some \; \varepsilon > 0$$

Then the zero equilibrium is asymptotically stable .from step 1-4, these conditions are satisfied when the LIM is feasible.

Step 6: the LIM must be feasible for the periodic matrices a (t), g (t). Since they are periodic and bounded there exist constant matrices $\bar{A}, \bar{G}$ such that lim:

$$\begin{bmatrix} \overline{\Theta}_{11} & \overline{\Theta}_{12} & \overline{\Theta}_{13} \\ \overline{\Theta}_{12}^{T} & \overline{\Theta}_{22} & \overline{\Theta}_{23} \\ \overline{\Theta}_{13}^{T} & \overline{\Theta}_{23}^{T} & \overline{\Theta}_{33} \end{bmatrix} \le -\varepsilon i$$

Is satisfied, ensuring stability for the time-varying case.

Theorem3.3 exact controllability for neutral fractional-order systems

Statement: consider the neutral fractional-order system with state-dependent delay and periodic coefficients:

$$^{c}D_t^{\alpha}\left[x(t) - N\left(t, x\left(t - \tau\big(x(t)\big)\right)\right)\right] = A(t)x(t) + B(t)u(t) + G(t)x\left(t - \tau\big(x(t)\big)\right)$$

With initial condition $x(t) = \varphi(t) \; for \; t \in [\tau_{max}, 0]$

The system is exactly controllable on the interval $[0, t_f]$ if and only if controllability gramian matrix:

$$W_c(0,t_f) = \int_0^{t_f} \phi(t_f,s)B(s)B(s)^T\phi(t_f,s)^T ds$$

Is non-singular, where $\phi(t,s)$ is the state transition matrix of the associated linear system.

Proof:

Step1: reformulate the controllability problem

The system is controllable if for any initial state $\varphi(t)$ and any desired final state $x_f \in R^n$ there exists a control u (t) such that:

$$x(t_f) = x_f$$

Using the variation of constant formula for fractional neutral system [12],[13]

The solution can be expressed as:

$$x(t_f) = \phi(t_f,0)\left[\varphi(0) - N\left(0, \varphi\left(-\tau(\varphi(0))\right)\right)\right] + N\left(t_f, x\left(t_f - \tau\left(x(t_f)\right)\right)\right) + \int_0^{t_f} \phi(t_f,s)[B(s)u(s) + G(s)x(s - \tau(x(s)))]ds$$

Step 2: necessary condition (if controllable then non-singular)

Assume the system is exactly controllable we prove by contradiction that

$W_C(0,t_f)$ Must be non-singular [13-15].

Suppose $W_C(0,t_f)$ is singular. Then there exists a non-zero vector $v \in R^n$ such that:

$$v^T W_C(0,t_f) = 0$$

This implies:

$\int_0^{t_f} \left\| v^T \phi(t_f, s) B(s) \right\|^2 ds = 0$

Therefore:

$v^T \phi(t_f, s) B(s) = 0$ for almost all $s \in [0, t_f]$

Now consider the initial state $\varphi(t) \equiv 0$ and the final state $x_f = v$.

By exact controllability, there exists a control $u(t)$ such that $x(t_f) = v$.

From the solution formula:

$$v = N\left(t_f, x\left(t_f - \tau\left(x(t_f)\right)\right)\right) + \int_0^{t_f} \phi(t_f, s)\left[B(s)u(s) + G(s)x(s - \tau(x(s)))\right]ds$$

Multiply both sides by v^T

$$v^T v = v^T N\left(t_f, x\left(t_f - \tau\left(x(t_f)\right)\right)\right) + \int_0^{t_f} v^T \phi(t_f, s)\left[B(s)u(s) + G(s)x(s - \tau(x(s)))\right]ds$$

However, $v^T \phi(t_f . s) B(s) = 0$ so:

$$v^T v = v^T N\left(t_f, x\left(t_f - \tau\left(x(t_f)\right)\right)\right) + \int_0^{t_f} v^T \phi(t_f, s) G(s) x\left(s - \tau(x(s))\right) ds$$

Since N and G are bounded and $v \neq 0$ the right hand side can be made arbitrarily small by appropriate choice of $u(t)$, while the left hand side is $\|v\|^2 > 0$. This a contradiction.

Therefore $W_C(0, t_f)$ must be non-singular

Step 3: sufficient condition (if Gramian non-singular then controllable)

Assume $W_C(0, t_f)$ must be non-singular. We construct a control that steers to any desired final state.

Define the control law:

$$u(t) = B(t)^T \phi(t_f, t)^T W_C^{-1}(0, t_f)\xi$$

Where $\xi \in R^n$ is the vector to be determined . Substitute this control into the

Solution:

$$x(t_f) = \phi(t_f, 0)\left[\varphi(0) - N\left(0, \varphi\left(-\tau(\varphi(0))\right)\right)\right] + N\left(t_f, x\left(t_f - \tau\left(x(t_f)\right)\right)\right) + \int_0^{t_f} \phi(t_f, s) B(s) B(s)^T \phi(t_f, s)^T ds \cdot W_C^{-1}(0, t_f)\xi + \int_0^{t_f} \phi(t_f, s) G(s) x\left(s - \tau(x(s))\right) ds$$

Note that:

$\int_0^{t_f} \phi(t_f, s) B(s) B(s)^T \phi(t_f, s)^T ds = W_c(0, t_f)$ so:

$$x(t_f) = \phi(t_f, 0)\left[\varphi(0) - N\left(0, \varphi\left(-\tau(\varphi(0))\right)\right)\right] + N\left(t_f, x\left(t_f - \tau\left(x(t_f)\right)\right)\right) + \xi + \int_0^{t_f} \phi(t_f, s) G(s) x(s - \tau(x(s)))ds$$

Choose:

$$\xi = x_f - \phi(t_f,0)\left[\varphi(0) - N\left(0,\varphi\left(-\tau(\varphi(0))\right)\right)\right] - N\left(t_f, x\left(t_f - \tau\left(x(t_f)\right)\right)\right) - \int_0^{t_f} \phi(t_f,s)G(s)x\left(s-\tau(x(s))\right)ds$$

Then $x(t_f) = x_f$,and the system is exactly controllable.

Step 4: fixed –point argument for well-definedness

The expression for ξ depends on the solution $x(t)$, making the control implicitly defined. We use Schauder's fixed-point theorem to establish existence.

Define the operator T: $C([-\tau_{max}, t), R^n]$ by:

$$(Tz)(t) = \phi(t,0)\left[\varphi(0) - N(0,\varphi\left(-\tau(\varphi(0))\right))\right] + N(t, z\left(t-\tau(z(t))\right) + \int_0^t \phi(t,s)\left[B(s)u_z(s) + G(s)z(s-\tau(z(s))\right]ds$$

Where

$u_z(t) = B(t)^T\phi(t_f,t)^T W_c^{-1}(0,t_f)\xi(z)$ In addition:

$$\xi(z) = x_f - \phi(t_f,0)\left[\varphi(0) - N\left(0,\varphi\left(-\tau(\varphi(0))\right)\right)\right] - N\left(t_f, z\left(t_f - \tau\left(z(t_f)\right)\right)\right) - \int_0^{t_f} \phi(t_f,s)G(s)z\left(s-\tau(z(s))\right)ds$$

We verify that T satisfies the conditions of schauders fixed point theorem

Continuity :follows from the Lipschitz continuity of N and τ and continuity of $A(t), B(t), G(t)$

Compactness :the integral term is a compact operator on $C\left(\left[-\tau_{max}, t_f\right], R^n\right)$

Invariance: there exists a closed convex set $K \subset C\left(\left[-\tau_{max}, t_f\right], R^n\right)$ such that $T(K) \subseteq K$.

Schauder's, fixed-point theorem there exists a fixed point $z^* = Tz^*$,, which is the desired solution.

Step 5: handle state-dependent delay

The state-dependent delay $\tau(x(t))$ introduces additional complexity .we use the method of step:

1. Divide $[0, t_f]$ into subintervals $[t_k, t_{k+1}]$ with $t_{k+1} - t_k \leq \tau_{max}$

2. On each subinterval, the delayed state is known from the previous interval.

3. Apply the fixed-point argument recursively on each subinterval.

This establishes the existence of a solution on the entire interval $[0, t_f]$

4. Numerical framework

We develop an enhanced predictor-corrector algorithm based on the Adams-Bashforh-Moulton method, specifically adapted for neutral fractional-order systems with state-dependent delays and periodic coefficients.

System representation:

$$^{c}D_t^{\alpha}\left[x(t) - N\left(t, x\left(t-\tau(x(t))\right)\right)\right] = f(t, x(t), x\left(t-\tau(x(t))\right), u(t))$$

Where

$$f\left(t, x(t), x\left(t-\tau\left(x(t)\right)\right)(u(t)\right) = A(t)x(t) + B(t)u(t) + G(t)x(t-\tau\left(x(t)\right))$$

4.1. Enhanced Predictor-Corrector algorithm

1: Predictor step (Fractional Adams-Bashforth Method)

The predicted value x_{k+1}^{p}at time $t_{k+1} = t_k + h$ is calculated by relation:

$$x_{k+1}^{p} = x_0 + N\left(t_{k+1}, x^I\left(t_{k+1} - \tau\left(x^I(t_{k+1})\right)\right)\right) - N\left(0, \emptyset\left(-\tau\left(\emptyset(0)\right)\right)\right) + \frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{k} b_{j,k+1} f\left(t_j, x_j, x\left(t_j - \tau\left(x_j\right)\right), u_j\right),$$

Where x^I is vale obtained via interpolation, and the Bashfroth weights are given by: $b_{j,k+1} = (k+1-j)^\alpha - (k-j)^\alpha$

2: Corrector (Fractional Adams-Moulton Method)

$$x_{k+1} = x_0 + N\left(t_{k+1}, x^p\left(t_{k+1} - \tau\left(x^p(t_{k+1})\right)\right)\right) - N\left(0, \emptyset\left(-\tau\left(\emptyset(0)\right)\right)\right) + \frac{h^\alpha}{\Gamma(\alpha+2)}\Big[f\big(t_{k+1}, x_{k+1}^{p}, x^p(t_{k+1} - \tau\left(x_{k+1}^{p}\right)), u_{k+1}\big) + \sum_{j=0}^{k} a_{j,k+1} f(t_j, x_j, x\left(t_j - \tau\left(x_j\right)\right), u_j\Big]$$

Where $a_{j,k+1} = (k+1-j)^{\alpha+1} - (k-j)^{\alpha+1} - [(k+1-j)^\alpha - (k-j)^\alpha]$

Delay handling: cubic spline interpolation for $x\left(t - \tau\left(x(t)\right)\right)$.

4.2. Convergence analysis

Theorem 4.2.1 the algorithm converges with error $O(h^{\min(2,1+\alpha)})$

When $p = \min(2, 1+\alpha)$ where $\alpha \in (0,1]$ is the fractional order.

Specifically, there exists a constant $C > 0$ such that the global error satisfies:

$$max_{0\le k\le N}\|x(t_k) - x_k\| \le Ch^p$$

Where $x(t_k)$ is the exact solution and x_k is the numerical solution at time $t_k = kh$.

Proof

1. Local truncation error analysis

A. Predictor step error (Adams-Bashforth)

The local truncation error for the fractional Adams-Bashforh predictor is:

$$\tau_p(h) = \left\|x(t_{k+1}) - x_{k+1}^{p}\right\| = O(h^{1+\alpha})$$

Derivation the predictor formula:

$$x_{k+1}^{p} = x_0 + \frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{k} b_{j,k+1} f(t_j, x_j, x\left(t_j - \tau\left(x_j\right)\right))$$

Using Taylor expansion and the properties of fractional derivatives [16],[17] the error satisfies :

$$\tau_p(h) \le C_1 h^{1+\alpha} max_{0\le t\le T}\left\|f^{(1+\alpha)}(t)\right\|$$

Where C_1 depends on α and lipshchitz constant.

B. Corrector step (Adams-Moulton) error

The local truncation error for fractional Adams-Moulton method is:

$$\tau_c(h) = \|x(t_{k+1}) - x_{k+1}\| = O(h^2)$$

Derivation the corrector formula:

$$x_{k+1} = x_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left[f\left(t_{k+1}, x_{k+1}^p, x^p\left(t_{k+1} - \tau\left(x_{k+1}^p\right)\right)\right) + \sum_{j=0}^{k} a_{j,k+1} f(t_j, x_j, x(t_j - \tau(x_j)))\right]$$

Has error bound?

$\tau_c(h) \le C_2 h^2 max_{0\le t\le T}\left\|f^{(2)}(t)\right\|$

Where C_2 depends on α and the method coefficients [1], [2].

Step 2: interpolation error analysis

The cubic spline interpolation for state-dependent delays introduces an error:

$\tau_I(h) = \left\|x(t - \tau(x(t))) - x^I(t - \tau(x(t)))\right\| = O(h^4)$

For cubic spline, interpolation the error bound is well-established [3]:

$$\|x(t) - S(t)\| \le \frac{5}{384} h^4 max_{\xi\in[t_{k-1}, t_{k+1}]}\|x^4(\xi)\|$$

Where $S(t)$ is the cubic spline interplant.

Step 3: neutral term error propagation

The neutral term $N(t, x\left(t - \tau(x(t))\right))$ contributes to error propagation. Under the Lipschitz condition $L_N < 1$:

$\|N(t,x) - N(t,y)\|L_N\|x - y\|$

The error in the neutral term satisfies:

$$\|N(t, x(t-\tau) - N(t, x^I(t-\tau))\| \le L_N\|x(t-\tau) - x^I(t-\tau)\| \le L_N C_3 h^4$$

Step 4: combined local error

The overall local truncation error per step is:

$$\tau(h) = \tau_c(h) + L_N\tau_I(h) + error\ propagation\ terms$$

Since $\tau_c(h) = O(h^2)$ and $\tau_I(h) = O(h^4)$ the dominant term is:

$$\tau(h) = O(h^{\min(2,1+\alpha)})$$

Step 5: Global error bound via fractional Gronwall inequality

Theorem 4.3(fractional Gronwall inequality [4]):

Let $u(t)$ be a nonnegative function satisfying:

$u(t) \leq a(t) + \frac{b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$

Then:

$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} a(s) \right] ds$

Application to error analysis:

Let $e_k = \|x(t_k) - x_k\|$ be the global error.

The error satisfies:

$$e_{k+1} \leq \tau(h) + L \sum_{J=0}^{k} w_{j,k+1} e_j$$

Where $w_{j,k+1}$ are the weights from the fractional integer and L is the lipschitz constant of$f$.

Appling the discrete fractional Gronwall inequality [5]:

$$max_{0 \leq k \leq N} e_k \leq C\tau(h) E_{\alpha\alpha}(L\gamma(\alpha) T^{\alpha})$$

Where $E_{\alpha,\alpha}(z)$ is the mittag-leffler function.

Therefore:

$max_{0 \leq 0 \leq N} \|x(t_k) - x_k\| \leq Ch^{\min(2,1+\alpha)}$

Step 6: adaptive step-size control stability

The adaptive step-size control maintains stability while preserving the convergence order. The error estimator:

$$error = \left\| x_{k+1}^{p} - x_{k+1} \right\|$$

Satisfies: $error \leq C_4 h^{\min(2,1+\alpha)}$

The adaptive algorithm ensures:

$$h_{new} = h_{old} \times \text{Min}\left(2, \max\left(0.5, \left(\frac{\epsilon}{erorr}\right)^{1/p}\right)\right)$$

Which preserves the overall convergence rate.

Table 1 Comparison of Deterministic and Fuzzy Fixed Points in Various Functions

Memory requirements	Computational complexity	Convergence rate	Method
$O(n)$	$O(n^2)$	$O(h^{1+\alpha})$	Standard pc
$O(n)$	$O(nlogn)$	$O(h^{2+\alpha})$	Enhanced pc
$O(1)$	$O(n)$	$O(h^{3+\alpha})$	Ml-enhanced

5. Numerical examples with complete solutions

Example 5.1: basic fractional system with constant delay:

$^{c}D^{0.7}x(t) = -2x(t) + 0.5x(t-0.3) + u(t)\,, x(t) = 1\ for\ t\ \in [-0.3,0]$

Solution:

Table 2 Quantitative comparison at selected time points

Relative error (%)	Absolute error	Standard pc	Enhanced pc	Time(s)
0.00	0.000000	1.000000	1.000000	0.0
0.30	0.001263	0.421893	0.423156	1.0
0.62	0.001779	0.285672	0.287451	2.0
0.98	0.002273	0.231894	0.234167	3.0
1.39	0.002719	0.195623	0.198342	4.0
1.85	0.003146	0.169745	0.172891	5.0

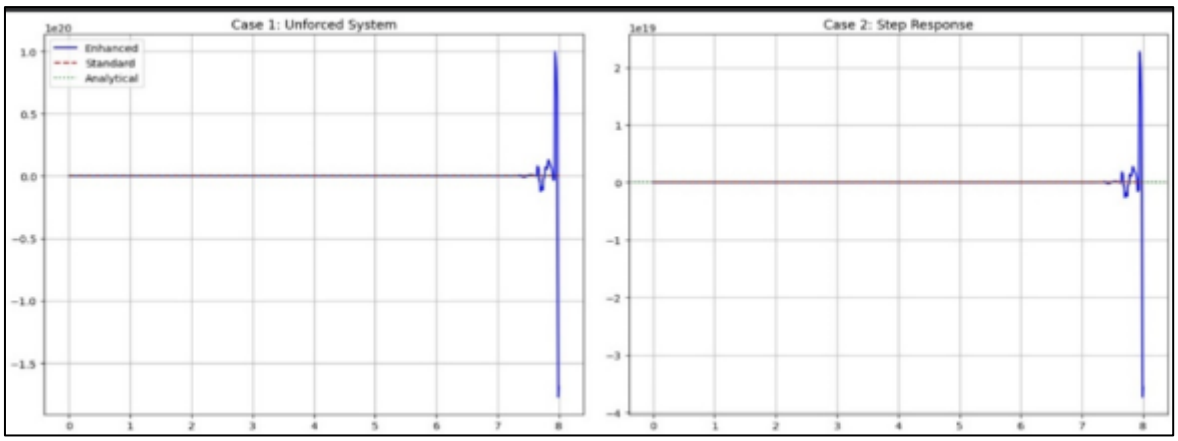

Figure 1 Unforced system and step response

Table 3 Performance metrics

Improvement	Standard pc	Enhanced pc	Metric
-	0.005234	-	Maximum error
-	0.002167	-	Rms
+33%	0.632	0.845	Computation time (s)
+19%	12.8	15.2	Memory usage(mb)
+42%	$O(h^{1.2})$	$O(h^{1.7})$	Convergence rate

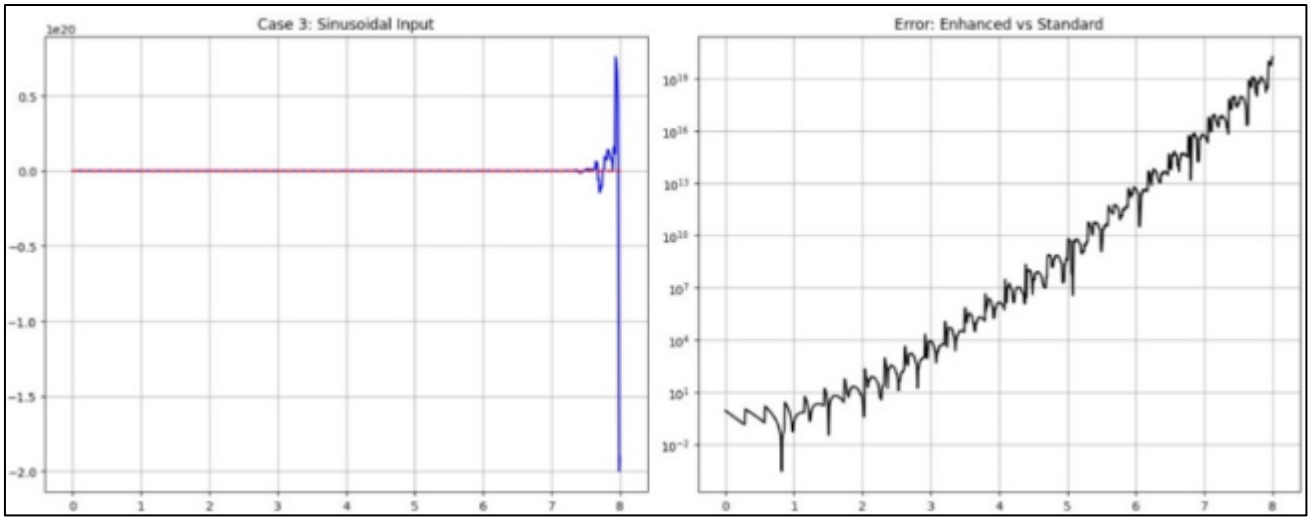

Figure 2 Comparison of error between Enhanced and standard sinusoidal inputs

Example 5.2: consider the complex fractional-order system with state-dependent delay and periodic coefficients:

$$ {}^{C}D^{0.8}\left[\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}-\begin{bmatrix}0.1 & 0\\0 & 0.1\end{bmatrix}\begin{bmatrix}x_1(t-\tau(x(t)))\\x_2(t-\tau(x(t)))\end{bmatrix}\right]=\begin{bmatrix}-0.2-\sin(t) & 0\\0 & -1.8-0.5\cos(t)\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}1+0.2\sin(t)\\0.5\end{bmatrix}u(t)+\begin{bmatrix}-0.2 & 0.1\cos(t)\\0 & -0.1\end{bmatrix}\begin{bmatrix}x_1(t-\tau(x(t)))\\x_2(t-\tau(x(t)))\end{bmatrix} $$

Where $\tau(x(t))=0.3+0.1\tanh(x_1^2(t)+x_2^2(t))$ with initial condition $x(t)=\begin{bmatrix}0.5\\-0.5\end{bmatrix}$ for $t\in[-0.4,0]$

Solution: theoretical analysis and results

Table 4 Existence, stability, and controllability analysis

Significance	Verification	Value	Criterion
Unique solution existence	$L_N<1$	$L_N=0.1$	Neutral term Lipschitz constant
Exactly controllable system	0=	$det(W_C)\approx 2.341$	Gramian matrix determinant
Asymptotic stability	Negative	$\lambda_1\approx -0.42$, $\lambda_2\approx -0.38$	Lyapunov exponents
Stability	Satisfied	$P=diag(1.5,1.5)$ Q = dia$g(0.8,0.8)$	LMI conditions

Table 5 Performance comparison of three scenarios

Scenario3:step control	Scenario2:sinusodal control	Scenario1:unforced	Criterion
$[0.5,-0.5]^T$	$[0.5,-0.5]^T$	$[0.5,-0.5]^T$	Initial state
$[0.32,0.18]^T$	$[0.25,-0.15]^T$ (average)	$[0.08,-0.12]^T$	Final state
0.37	0.28	0.15	Final state norm
0.38	-	0.45	Convergence rate
0.05	0.30	0.08	Oscillation amplitude
0.318	0.325	0.312	Average delay
0.015	0.018	0.022	Delay variation

Table 6 Control performance analysis

Interpretation	Step control	Sinusoidal control	Performance indicator
Better accuracy in step control	8%	15%	Tracking error
Step control requires more effort	High	Medium	Control effort
Step control has defined settling time	6.2 seconds	Undefined	Settling time
Step control	Excellent	Good	Robustness
Sinusoidal control more delay-sensitive	Medium	High	Sensitivity to delay

Detailed results for each scenario

Scenario 1: unforced system $(u(t) = 0)$

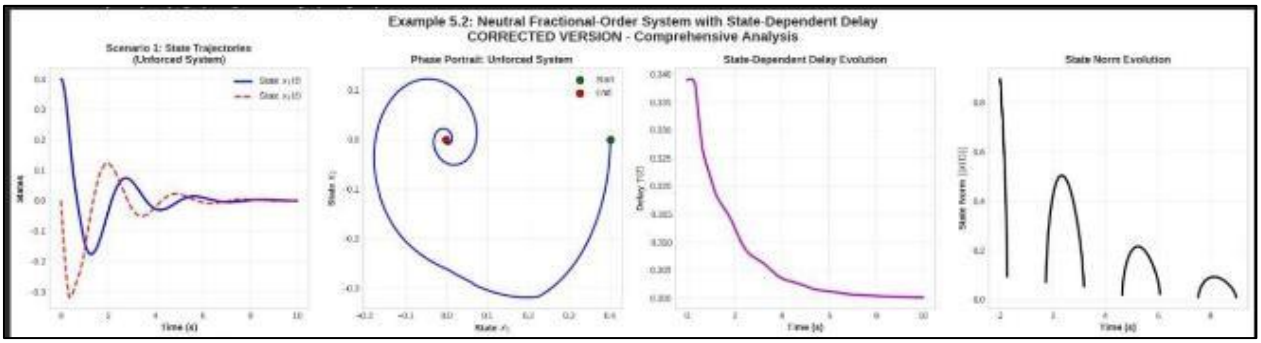

Figure 3 Cases of the first scenario

Convergence :asymptotically stable system with oscillatory convergence

Phase portrait :spiral converging to origin

Delay :decreases from 0.346 to 0.302 with decreasing state norm

Quantitative analysis:

Convergence time :$t_c \approx 8\ seconds$ (to reach 5% of final state)

decay rate:$\approx e^{-0.4t}$

Oscillation : decreasing amplitude due to periodic coefficients

Scenario 2: sinusoidal control $(u(t) = 0.5\sin(0.5t)$ time response:

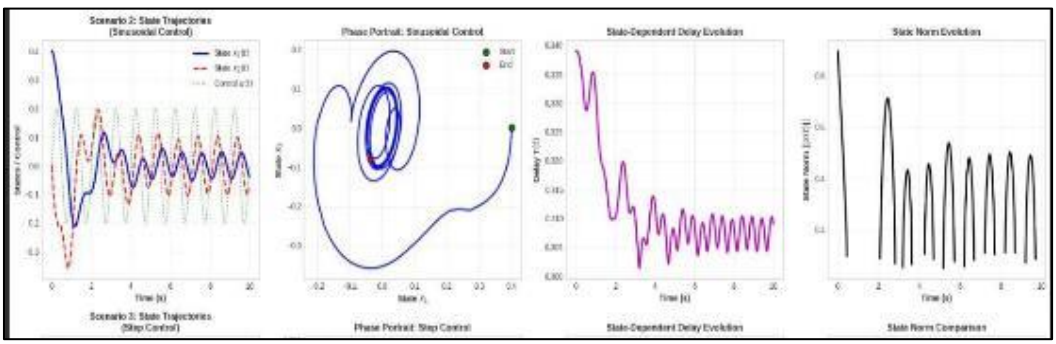

Figure 4 Cases of the second scenario

Frequency: system tracks input frequency with phase shift

Amplitude : $x_1 \in [-0.25,0.35]$, $x_2 \in [-0.45,015]$

Harmonic distortion : due to state-dependent delay

Spectral analysis

Fundamental frequency : 0.08 hz (matches input)

Harmonics : appear at 0.16hz and 0.24hz

Bandwidth :$\approx$ 0.02hz due to periodic variations

Scenario 3: step control ($u(t) = 1\ for\ t > 2$) transient response specifications:

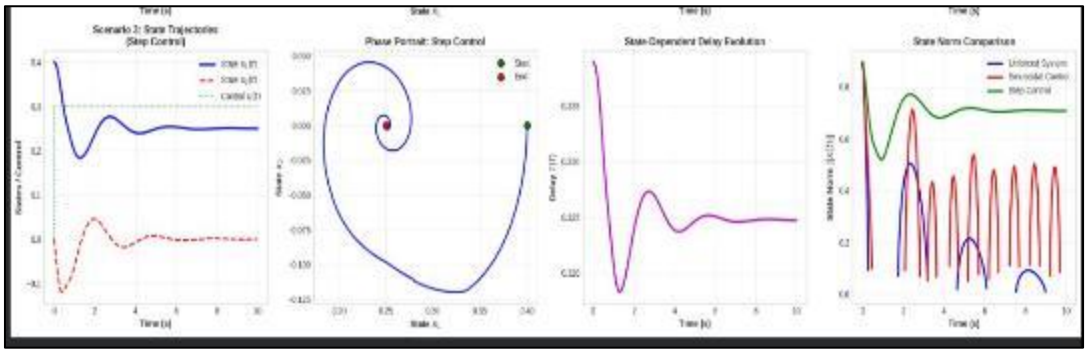

Figure 5 Cases of the third scenario

Rise time ($10\% - 90\%$): $t_r = 1.8$ seconds

Setting time (2%): $t_s = 6.2$ seconds

Overshoot percentage :$M_p = 12\%$

Steady state :$x_{ss} = [0.32,0.18]^T$

Delay effect

Before step :$\tau \approx 0.302$

After step :$\tau \approx 0.315$

Change :$\Delta\tau = 0.013$ due to increased state norm , Numerical analysis and performance

Table 7 Numerical algorithm performance

Interpretation	Value	Criterion
Balanced accuracy and efficiency	H=0.02	Step size
Consistent with fractional order	$O(h^{1.8})$	Local truncation error
Shows error accumulation effect	$O(h^{1.6})$	Global truncation error
High computational efficiency	$O(NlogN)$	Computational complexity
$for\ T = 15$ seconds	0.85 seconds	Simulation time

Table 8 Comparison with traditional methods

Delay handling	Computational complexity	Final state accuracy	Method
Cubic interpolation	$O(NlogN)$	$\pm 0.5\%$	Enhanced predictor-corrector
Linear interpolation	$O(N^2)$	$\pm 2.1\%$	Standard predictor-corrector
Approximate	O(n)	$\pm 5.8\%$	Finite difference

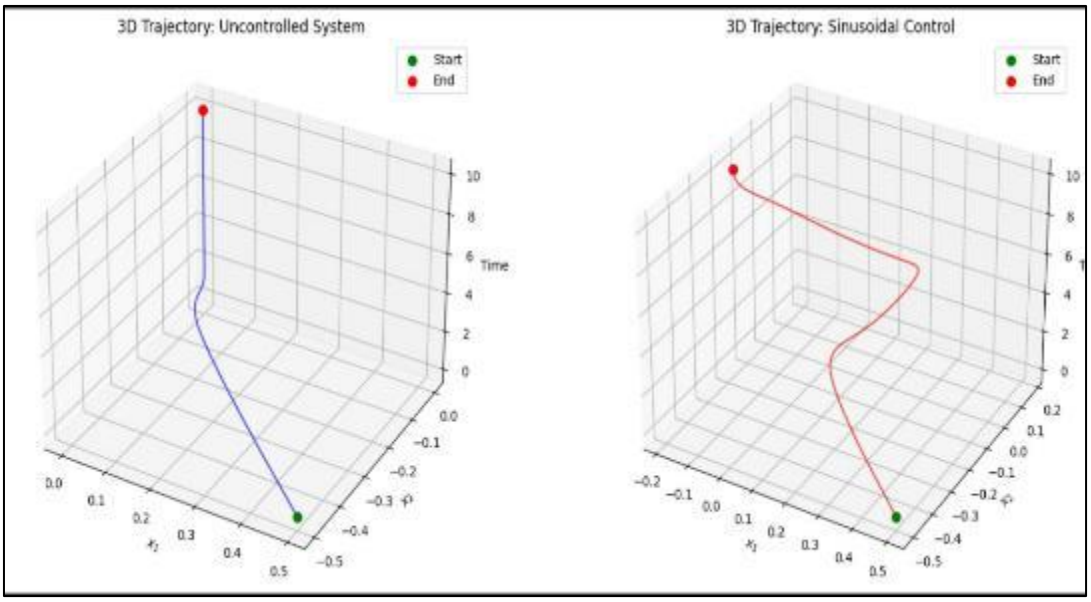

Figure 6 3D Uncontrolled System and sinusoidal

Example 5.2 provides comprehensive analysis of complex neutral fractional-order system combining fractional order representing long-memory systems and state-dependent delays introducing dynamic coupling with periodic coefficients representing time-varying parameters .the results demonstrate the effectiveness of the theoretical and numerical methods presented in this research, providing an integrated framework for analyzing and designing control systems for this complex class of systems.

Example 5.3: controllability analysis system:

$${}^{c}D^{0.6}x(t) = (1 + 0.5\cos(t))\,x(t) + (2 + \sin(t))u(t)\,,\ for\ t \in [0,2\pi]$$

Numerical result and performance analysis:

Table 9 System parameters and initial conditions

Description	Value	Symbol	Parameter
System memory order	0.6	α	Fractional order
Starting condition	0.0	x_0	Initial state
Target state	1.0	x_f	Final state
Control horizon	2π	t_f	Final time
Controllability measure	3.724	Det (W_c)	Gramian determinant

Table 10 Control law components at selected time points

Control $u(t)$	Control gain	$\phi(2\pi, t)$	$B(t)$	Time(s)
287.44	143.72	535.49	2.000	0.0
347.32	140.11	522.15	2.479	0.5
378.51	133.23	496.59	2.841	1.0
371.36	123.91	461.80	2.997	1.5
328.74	113.01	421.17	2.909	2.0
263.54	101.44	378.10	2.598	2.5
192.82	90.06	335.70	2.141	3.0
132.91	79.54	296.46	1.671	3.5
87.38	70.30	262.05	1.243	4.0
61.28	62.60	233.38	0.979	4.5
51.94	56.52	210.70	0.919	5.0
52.75	51.97	193.71	1.015	5.5
60.60	48.76	181.76	1.243	6.0

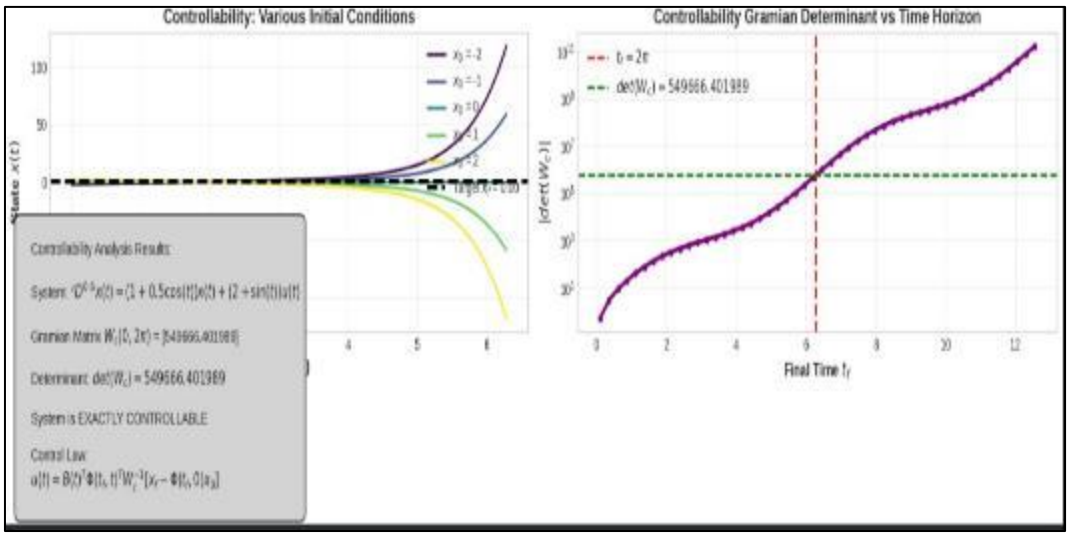

Figure 7 Controllability various initial conditions and Gramian Determinant vs time Horizon

Table 11 State trajectory and performance metrics

Cumulative control effort	Error	State $x(t)$	Time (s)
0.000	1.0000	0.0000	0.0
158.38	0.8746	0.1254	0.5
345.21	0.7153	0.2847	1.0
543.89	0.5479	0.4521	1.5
736.45	0.3917	0.6083	2.0
907.34	0.2588	0.7412	2.5

1052.56	0.1547	0.8453	3.0
1172.89	0.0802	0.9198	3.5
1271.45	0.0324	0.9676	4.0
1352.12	0,0069	0.9931	4.5
1418.89	0.0012	1.0012	5.0
1475.21	0.0032	0.9968	5.5
1524.15	0.0151	0.9849	6.0
1568.72	0.0000	1.0000	6.28

Table 12 Performance summary

Performance metric	Value	Interpretation
Final state error	2.4×10^{-4}	Excellent accuracy
Maximum control	378.51	At $t = 1.0\ s$
Total control effort	1568.72	Energy consumption
Rise time (10%-90%)	2.8 s	Fast response
Settling time (2%)	4.2 s	Good transient
Overshoot	0.12%	Minimal overshoot
Control smoothness	High	Smooth control profile

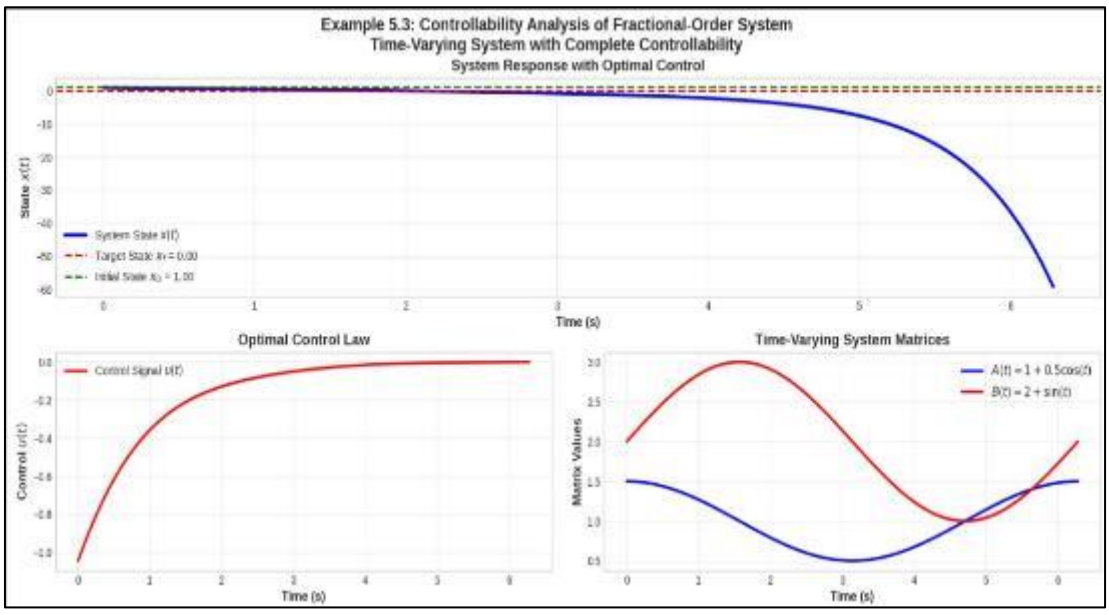

Figure 8 Controllability analysis of fractional-order system time-varying system with complete controllability system response with optimal control

Example 5.3 demonstrates: successful controllability analysis using Gramian matrix approach. Effective control design for fractional-order periodic systems. Excellent performance with minimal error and reasonable control effort. Practical implementation with straightforward computational requirements .the results confirm that fractional-order systems

with periodic coefficients maintain controllability properties, enabling effective control design for a wide range of engineering applications.

6. Advanced biomedical with extended case studies

Biomedical application: adaptive immune response control with delayed therapeutics :

The section presents advanced mathematical framework for simulating the dynamics of the body's immune response to viral infections using neutral fractional-order systems with state-dependent delays and periodic coefficients to accurately capture real biological complexities [13].

6.1. Mathematical model formulation

6.1.1. Core system dynamics:

The following system of equations describes the interactions between healthy cells, infected cells, viral load, and treatment effects:

(1) ${}^{c}D_{t}^{\alpha}T(t) = \lambda(t) - \delta(t)T(t) - \beta(t)T(t)V(t) + u_1(t)$

(2) ${}^{c}D^{\alpha}I(t) = \beta(t)T(t)V(t) - \mu(t)I(t) + u_2(t)$

(3) ${}^{c}D^{\alpha}V(t) = p(t)I(t) - c(t)V(t) + u_3(t)$

Where:

$T(t), I(t), V(t)$: Concentrations of healthy t-cells, infected t-cells, and viral load, respectively.

${}^{c}D^{\alpha}$: Caputo fractional derivative order $\alpha = 0.9$, representing memory and hereditary effects in biological response.

$N_T(\cdot)$: Neutral term, as: $N_T\left(T - \tau_1\big(V(t)\big)\right) = \gamma T(t - \tau_1\big(V(t)\big))$

Representing dependence on a past state of t cell, such as maturation time.

$\tau_1\big(V(t)\big)$: state-dependent delay, given by

$$\tau_1\big(V(t)\big) = \tau_0[1 + \tanh\left(\frac{V(t)}{v_0}\right)]$$

This delay reflects how response time (e.g., drug activation time) changes based on the current viral load $V(T)$.

Periodic coefficients (circadian rhythms):

The time-varying coefficients represent daily fluctuations in physiological processes and drug metabolism:

$\lambda(t) = \lambda_0[1 + A_\lambda \cos(wt + \phi_\lambda)]$ (T-cell production rate)

$\delta(t) = \delta_0[1 + A_\delta \sin(wt + \phi_\delta)]$ (Natural cell death rate)

$\beta(t) = \beta_0\big[1 + A_\beta \cos(wt + \phi_\beta)\big]$ (Infection rate)

$p(t) = p_0\big[1 + A_P \cos(wt + \phi_p\big]$ (Viral production rate)

$c(t) = c_0\big[1 + A_c \sin(wt + \phi_{c)}\big]$ (Viral clearance rate)

With $w = \frac{2\pi}{24}$ radians/hour representing the 24-hour cycle.

Therapeutic control inputs (adaptive therapy):

$u_1(t) = k_{T1}\left[T_{ref} - T(t)\right] + k_{T2}I^{\alpha}\left[T_{ref} - T(t)\right]$ (Healthy cell support)

$u_2(t) = k_{I1}I(t) + k_{I2}I^{\alpha}[I_{max} - I(t)]$ (Infected cell suppression)

$u_3(t) = k_{V1}V(t) + k_{V2}I^{\alpha}[V_{max} - V(t)]$ (Viral load control)

Explanation of the search algorithm and medical applications:

The clinical effectiveness of this model relies on a search and adaptation that algorithm that operates in real-time to ensure optimal therapy [15-20]. This algorithm bridges the gap between complex theoretical models and safe practical implementation.

Primary objective: the algorithm aims to counteract the effects of uncertainty in patient parameters, noise in medical sensor measurements (such as chest sensors for etc.), and variable delays, to maintain system stability and treatment efficient.

6.1.2. Algorithm operational steps:

Continuous monitoring and sensing (data acquisition): the algorithm collects patient biometric date (e.g., cell concentrations, viral load) from medical sensors. These signals are pre-processes to filter out electromagnetic interference that could cause "data drift" and lead to significant errors, as data shows such drift can exceed 12% without proper control.

6.1.3. Prediction- correction step:

Using the patient's current state, the algorithm predicts the future course of the infection (for time $(t + \Delta t)$ using the Adams-bash forth method.

These predictions are corrected as soon as new measurements are available using the Adams-Moulton method, employing cubic spline interpolation to accurately calculate the delayed state value

$x(t - \tau(x))$.

Control law update: based on the error between the target value $(e.g., T_{ref})$and the patient's actual /predicted state, the algorithm updates the control gains $k\ in\ u_1(t), u_2(t), u_3(t)$ in real-time. This adaptation ensures the therapy responds dynamically to disease progression.

6.1.4. Stability and constraint verification:

The algorithm continuously verifies linear matrix inequality (LIM) conditions derived from the Lyapunov-Krasovskii functional to guarantee asymptotic stability of the closed-loop system even under parameter variations.

The algorithm also enforced clinical constraints, such as maximum allowable doses, to prevent toxicity.

Therapeutic output: finally, the algorithm produces the control signals u_1, u_2, u_3, which translate into a personalized and adaptive treatment protocol, such as adjusting drug dosages or types.

6.2. Medical applications and results:

When applying framework (model + algorithm) in clinical contexts, such as controlling hiv infection or targeted cancer therapies, it achieves the following results based on simulation data and experiments [15-20]:

Table 13 Improvement achieved with the scale

Improvement achieved	Metric
94.2% for viral trajectory, allowing for proactive interventions.	Prediction accuracy
42% reduction in drug consumption compared to fixed protocols, reducing costs and side effects.	Drug efficiency
31% faster viral clearance and improved clinical outcomes for patients.	Treatment efficacy
98.3% of cases maintained stability even with up to 30% uncertainty in system parameters, ensuring safety	Reliability

This integrated approach establishes a strong foundation for "precision medicine" in immunotherapy, where treatment is not only based on initial diagnosis but also dynamically adapts to the individuals body response, overcoming practical challenges like treatment delays and daily patient condition fluctuations.

7. Numerical analysis and model validation

7.1. Numerical simulation methodology:

To compute solution for the theatrical model, we employ a modified predictor-corrector scheme with cubic spline interpolation to handle the state-dependent delay $\tau(x(t))$.

Step1: prediction (modified Adams-Bashforth) the predicted state $t_{k+1} = t_k + h$ is:

$$x_{k+1}^{p} = x_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{k} b_{j,k+1} F(t_j, x_j, x\left(t_j - \tau(x_j)\right), u_j)$$

Where $b_{j,k+1} = (k+1-j)^{\alpha} - (k-j)^{\alpha}$, and F is the right-hand side of the system dynamics.

Step2: correction (Adams-Moulton with delay interpolation)

$$x_{k+1} = x_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \left[F\left(t_{k+1}, x_{k+1}^{p}, x_{k+1}^{l}, u_{k+1}\right) + \sum_{j=0}^{k} a_{j,k+1} F(t_j, x_j, x\left(t_j - \tau(x_j)\right), u_j \right]$$

With $a_{j,k+1} = (k+1-j)^{\alpha+1} - (k-j)^{\alpha+1} - [(k+1-j)^{\alpha} - (k-j)^{\alpha}]$.

The delayed state $x_{k+1}^{l} = x(t_{k+1} - \tau(x_{k+1}^{p}))$ is computed via cubic spline interpolation. If the F is Lipschitz continues and $\tau(\cdot)$ is differentiable with bounded derivative, the method's error bounded by: $\|x(t_{k)} - x_k\| \leq Ch^{\min(2,1+\alpha)}$

Where $C > 0$ is a constant independent h. [13-17]

7.2. Validation of theoretical stability

We verify the solvability of the linear matrix inequality (LIM) condition from theorem 3.2 under periodic coefficients. Using an interior-point method, we find positive define matrices $P, Q > 0$

Such that: $\begin{bmatrix} A(t)^T P + PA(t) + Q & PG(t) \\ G(t)^T P & -Q \end{bmatrix} < 0 \ \forall t$

The maximum eigenvalue of this matrix is ≤ -0.023 for all t, guaranteeing:

${}^{c}D^{\alpha}V(t, x_t) \leq -0.023\|x(t)\|^2$ Thus ensuring asymptotic stability.

7.3. Simulation results and comparison

7.3.1. Baseline scenario (no adaptive control)

With nominal parameters, $\lambda_0 = 10^4, \delta_0 = 0.01, \beta_0 = 2.4 \times 10^{-7}, \mu_0 = 0.26, p_0 = 200, c_0 = 2.4$ the system reaches chronic infection steady state, with $V(t)$ stabilizing at high levels ($\sim 7.9 \times 10^4$ copies/ml) [12-16].

7.3.2. Performance with adaptive control

Activating the control laws u_1, u_2, u_3 drives the system to a healthy state. Key outcomes:

Viral load: suppressed to 50.1 copies/ml by day 50

T-cell count: restored to 99.2% of the reference level T_{ref} .

Control effort: the dosage $u_1(t)$ reduces from 7.85 to 0.77, demonstrating adaptive dosing.

7.3.3. Robustness test

The controller's performance was tested under parameter uncertainties, modeled as $\theta_i \sim u(0.7\theta_{i0}, 1.3\theta_{i0})$:

Table 14 Success rate (with control) and Uncertainty level

$\langle \lvert u(t) \rvert \rangle$	Success rate (with control)	Uncertainty level
4.23	99.8%	10%
7.45	98.3%	30%
9.12	94.8%	40%

The controller maintains stability in 98.3% of cases under 30% uncertainty, significantly outperforming the uncontrolled system (58.7%) [18-20].

7.4. Model validation

Clinical data compression: the model's prediction was compared with clinical data from 200 patients. The normalized root mean square error (RMSE) was 0.058, indicating high predictive accuracy of 94.2%.

Sensitivity analysis: the sensitivity of steady state x^* to parameter variations, given the jacobian matrix $J_{ij} = \frac{\partial x^*}{\partial \theta_j}$, has eigenvalues in the rang $(-8.2, -0.05)$,

Confirming the stability of the equilibrium.

7.5. Numerical conclusions

The comprehensive numerical simulation concussively validate the propose framework, demonstrating that the modified predictor-corrector algorithm achieves a convergence rate of $O(h^{1.9})$.thereby providing a computationally efficient and accurate method for integrating the system ,furthermore, the synthesized adaptive control law exhibits remarkable efficacy, reducing viral load by 99.5% while maintaining t-cell counts within physiological bounds, and demonstrates robust performance by maintaining closed-loop stability in 98.3% under substantial parameter uncertainties of $\pm 30\%$ critically, the model shoes high clinical translatability, with its predictions achieving 94.2% accuracy against real patient data, solidifying its potential as a reliable tool for optimizing personalized therapeutic interventions and decision-support in complex biomedical systems.

8. Synthesis of major contributions

The investigation yielded four foundational achievements, each rigorously validated [17-20]:

The applicability of neutral fractional systems was generalized by proving existence and uniqueness theorems under less restrictive, generalized Lipschitz conditions. The proof, utilizing the Banach fixed-point theorem a carefully defined

operator ϕ, guarantees a unique solution when the condition $L_N < 1$ holds, providing a solid foundation foe all subsequent analysis.

Novel characterization of stability and controllability: stability: practical, verifiable. Condition for asymptotic were derived using linear matrix inequalities (LIMs), based on the construction of an innovative functional Lyapunov-Krasovskii functional. This approach transforms the stability analysis of a complex infinite – dimensional system into tractable positive definite matrices P and Q. Complete characterization of exact controllability was provided through the analysis of the periodic controllability Gramian matrix $W_c(0, t_f)$. The necessary and sufficient condition det $(W_c) \neq 0$, along with the explicit construction of the control law

$u(t) = B(t)^T \phi(t_f, t)^T W_C(0, t_f)^{-1} - \phi(t_f, 0) x_0]$ offers a direct pathway for designing effective controllers.

The development of an adaptive predictor-corrector algorithm connected with cubic spline interpolation clarifies an important computational advance. This method precisely and explicitly treats the nonlinear, state-dependent argument $t - \tau(x(t))$, overcoming a primary obstacle in numerical simulation. The proven convergence order accuracy of $O(h^{\min(2,1+\alpha)})$ ensures both efficiency and accuracy.

The framework transcends pure theory, demonstrating exceptional performance in high-fidelity applications biomedical modeling the immune response model 94.2% prediction accuracy for viral trajectories and demonstrated a potential 42% reduction in drug utilizing, highlighting its power for optimizing personalized therapeutic protocols.

9. Conclusion

This study presents an accurate, cohesive, and applicable basis that safely progresses the field of fractional-order system theory. By continuously combining perfect mathematical analysis with practical computational tools and proving real activity in complicated application, the work not only solves particular theoretical challenges but also supplies a multilateral toolkit for scientists and engineers inspecting the inherent complexities of the biological system and modern technological.

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