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The Effect of Fuzziness on the Stability of Fixed Points between Certainty and Probability

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Abstract

This paper is the research on how fuzziness may be incorporated in classical fixed-point theory. Fixed-point iteration is a basic numerical algorithm to solve equations of the form x= g(x). Nevertheless, in the real world, there is usually uncertainty, which deterministic models cannot represent. With the addition of the fuzzy parameters into the governing functions, fixed points that exist as a single exact value change to stability intervals. The study illustrates using analytic and quantitative examples of algebraic functions (linear, quadratic, trigonometric, and radical) the way in which the nature of fixed points changes through fuzziness. The results show that the linear systems maintain constant periods even in small perturbations of the system with fuzzy values, whereas nonlinear systems can have considerable topological variations, such as the disappearance of the fixed points. The work connects the theory of fixed-point and fuzzy logic, and provides a stronger mathematical framework on modeling uncertain dynamical systems, and has found use in control theory, economics, and artificial intelligence.

Keywords: Fixed point; Fuzzy Logic; Stability; Uncertainty; Numerical Methods; Iteration; Fuzzy Metric Spaces; Nonlinear Systems

1. Introduction

Fixed-point theory is an important branch of numerical analysis and of functional analysis, which offers fundamental methods of equation solution, of behavioural modeling of equilibrium states, and of dynamical systems analysis [1]. The classical iteration scheme $x_{n+1}=g(x_n)$ converges to the fixed point $x^*=g(x^*)$ provided that $g(x^*)$ has a fixed point with by the Banach Fixed-Point Theorem [2] , namely $g(x^*)$ that satisfies $|g'(x^*)|<1$ .Nevertheless, most physical, economic, and engineering systems have never been crisp and deterministic and must be left to be uncertain or vague.

Fuzzy logic which was developed by Zadeh [3] is an extension of classic binary logic and these degrees of truth are expressed as membership functions between 0 and 1. It has been found to be effective in dealing with imprecision especially in control systems, decision-making, and artificial intelligence [4]. Recent work has started to consider the intersection between the fixed-point theory and fuzzy sets, specifically in fuzzy metric spaces [5,6].

In this paper the role of the introduction of fuzziness as a bounded perturbation μ on the existence, uniqueness, and stability of fixed points are discussed. By using a succession of mathematical examples, we show how deterministic fixed points grow into intervals and this is indicative of the uncertainty of the system. This research is an addition to the current one of integrating the fuzzy logic in the classical analysis and was able to provide insights into how systems would be in a state of uncertainty.

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2. Methodology: Theoretical Framework and Practical Aspects.

This section gives the classical fixed-point iteration technique, gives the concept of fuzzy logic and shows how they can be integrated with real-life examples.

Classical fixed-point iteration refers to a solution-seeking procedure carried out by an iterative process.

2.1. Classical Fixed-Point Iteration.

Classical fixed-point iteration is a solution-seeking process implemented using an iterative process.

To a function g: R→R a fixed point x^* , x=g(x) . The iterative scheme:

$x_{n+1}=g\ (x_n)$

Approaches x, in the event that $|g'(x^*)|<1$,first in a guess, in case we are making a guess that is close enough to x. The meeting is linear and the error vanishes proportionally to $|g'(x^*)|$ [7].

Example: Solving cos(x)=x

Reformulated as x=cos(x) and g(x) = cos(x), g' (x) = -sin(x) At the fixed point $x^* \approx 0.739$, $|g'(x^*)| \approx 0.673<1$ ensuring convergence.

2.2. Fuzzy Logic and Fuzziness

Fuzzy logic is concerned with approximate, instead of fixed and precise reasoning. An additive uncertain parameter $\mu_A(x)\in[0,1]$, epsilon can be used to introduce fuzziness to the mathematical functions in a fuzzy set A, where $\mu\in[-\varepsilon,\varepsilon]$is defined as the degree to which x belongs to A.

2.3. Admission of Fuzziness in Fixed-Point Equations.

Take an example of a deterministic function f(x). The fuzzy version of it can be defined as:

$f_{\text{fuzzy}}(x)=f(x)+\mu, \quad \mu\in[-\varepsilon,\varepsilon].$

The equation of the fixed point is:

$x=f(x)+\mu.$

A solution to this gives a fuzzy fixed point which is often represented as an interval $x\in[a,b]$, instead of a value.

3. Results and Discussion

It is shown that four representative examples are analyzed to illustrate the impact of fuzziness.

3.1. Example 1: Linear Function

Deterministic:

f(x)=0.6x+2

Solving x=0.6x+2 gives x=5.

Fuzzy:

$f_{\text{fuzzy}}(x)=0.6x+2+\mu, \quad \mu\in[-0.1,0.1]$

Solving $x=0.6x+2+\mu$ yields : $x=(2+\mu)/0.4 = (5+2.5\mu)$. For $\mu\in[-0.1,0.1]$,

the fuzzy fixed point is: $x\in[4.75, 5.25]$.

3.2. Example 2: Quadratic Function

Deterministic:

$f(x) = x^2-2x+2$

Fixed points satisfy $x = x^2-2x+2$, i.e., $x^2-3x+2 = 0$ Solutions: $x = 1$ and $x = 2$.

Fuzzy:

$f_{fuzzy}(x)=x^2-2x+2+\mu, \quad \mu\in[-0.2,0.2]$

Equation: $x^2-3x+2+\mu=0$

Using the quadratic formula: $x= (3\pm\sqrt{1-4\mu})/2$.

For $\mu=-0.2$: $x \approx (3\pm1.34)/2 \Rightarrow x\in [0.83, 2.17]$

For $\mu=0.2$: $(\sqrt{1-0.8}) \approx 0.45 \Rightarrow x \approx (3\pm0.45) /2 \Rightarrow x\in [1.275, 1.725]$

Therefore, fuzziness has the ability to blend or move fixed-point ranges.

3.3. Example 3: Trigonometric Function

Deterministic:
$f(x)=\cos(x)$
Fixed point: $x^* \approx 0.739085$

Fuzzy:
$f_{fuzzy}(x)= \cos(x)+\mu, \quad \mu\in[-0.05, 0.05]$

The fixed point becomes an interval around x^*, approximately: $x\in[0.689,0.789]$.

3.4. Example 4: Square Root Function:

This example will be a calculation of the square root of a number.

Deterministic:

$f(x)=(\sqrt{x+3})$

Solving $x=(\sqrt{x+3})$, yields $x \approx 2.30278$

Fuzzy:

$f_{fuzzy}(x)=(\sqrt{x+3})+\mu, \quad \mu\in[-0.1,0.1]$

Solving $(x-\mu)^2= x+3$ and evaluating for $\mu=\pm0.1$ gives: $x\in[2.15, 2.43]$.

The findings are always that the fuzziness converts sharp fixed points into intervals. Key observations include:

1. Linear Stability: In linear functions (Example 1), when there is a small fuzzy perturbation then it still remains stable in a bounded sense, in a proportional interval around a deterministic fixed point.

2. Nonlinear Sensitivity Quadratic and transcendental functions (Examples 2-4) are more complicated. Fuzziness can:

- Reduce several fixed points to a continuum.
- Nonlinearly shift the interval.
- In others, cause the blowout of fixed points when the discriminant turns negative.

3. Implications to Theory: To generalize the Banach Fixed-Point Theorem to fuzzy metric spaces, the definition of contraction mappings must be redefined in terms of membership degrees, instead of crisp distances [8,9]. Examples of our relevancy to the recent research on fuzzy contractions and its fixed points [10].

4. Real-world Relevance: In control systems a fuzzy fixed-point interval is a stability region, as opposed to an exact setpoint, which increases resistance to noise and parameter drift. This comes in handy especially in adaptive fuzzy controllers and economic models that have uncertain parameters [11,12].

4. Conclusions

This work shows that representation of fuzziness in the theory of fixed-point can transform essence of solutions:

- Single values are developed into fixed points, which are more true to reality.
- It is the linearity of the function that determines how stable a fixed point is when faced with fuzziness, nonlinear systems can undergo qualitative behavior.
- Fuzzy fixed-point theory gives a more accommodating approach to systems modeling where probability, vagueness, and uncertainty are all present.

The addition of fuzzy logic to fixed-point analysis would be a great leap towards having more realistic mathematical modeling that skips the gap between accurate theory and inaccurate reality.

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